THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as Second-class Mail Matter.

VOL. IV. DECEMBER, 1897. No. 12.

BIOGRAPHY.

LEONHARD EULER.

BY B. F. FINKEL.

LEONHARD EULER (oi'ler), one of the greatest and most prolific mathematicians that the world has produced, was born at Basel, Switzerland, on the 15th day of April, 1707, and died at St. Petersburg, Russia, November the 18th (N. S.), 1783. Euler received his preliminary instruction in mathematics from his father who had considerable attainments as a mathematician, and who was a Calvinistic* pastor of the village of Riechen, which is not far from Basel. He was then sent to the University of Basel where he studied mathematics under the direction of John Bernoulli, with whose two sons, Daniel and Nicholas, he formed a life-long friendship. Geometry soon became his favorite study. His genius for analytical science soon gained for him a high place in the esteem of his instructor, John Bernoulli, who was at the time one of the first mathematicians of Europe. Having taken his degree as Master of Arts

**The Encyclopedia Brittanica* says Euler's father was a Calvinistic minister, while W. W. R. Ball, in his *History of Mathematics*, says he was a Lutheran minister. Euler himself was a Calvinist in doctrine, as the following, which is his apology for prayer, indicates: "I remark, first, that when God established the course of the universe, and arranged all the events which must come to pass in it, he paid attention to all the circumstances which should accompany each event; and particularly to the dispositions, to the desires, and prayers of every intelligent being; and that the arrangement of all events was disposed in perfect harmony with all these circumstances. When, therefore, a man addresses God a prayer worthy of being heard it must not be imagined that such a prayer came not to the knowledge of God till the moment it was formed. That prayer was already heard from all all eternity; and if the Father of Mercies deemed it worthy of being answered, he arranged the world expressly in favor of that prayer, so that the accomplishment should be a consequence of the natural course of events. It is thus that God answers the prayers of men without working a miracle."

in 1723, Euler afterwards applied himself, at his father's desire, to the study of theology and the Oriental languages, with the view of entering the ministry, but, with his father's consent, he returned to his favorite pursuit, the study of mathematics. At the same time, by the advice of the younger Bernouillis, who had removed to St. Petersburg in 1725, he applied himself to the study of physiology, to which he made useful applications of his mathematical knowledge; he also attended the lectures of the most eminent professors of Basel. While he was eagerly engaged in physiological researches, he composed a dissertation on the nature and propagation of sound. In his nineteenth year he also composed a dissertation in answer to a prize-question concerning the masting of ships, for which he received the second prize from the French Academy of Sciences.

When his two close friends, Daniel and Nicholas Bernoulli, went to Russia, they induced Catherine I, in 1727, to invite Euler to St. Petersburg, where Daniel, in 1733, was assigned to the chair of mathematics. Euler took up his residence in St. Petersburg, and was made an associate of the Academy of Sciences. In 1730 he became professor of physics, and in 1733 he succeeded his friend Daniel Bernoulli, who resigned on a plea of ill health.

At the commencement of his astonishing career, he enriched the Academical collection with many memoirs, which excited a noble emulation between him and the Bernouillis, though this did not in any way affect their friendship. It was at this time that he carried the integral calculus to a higher degree of perfection, invented the calculation of sines, reduced analytical operations to greater simplicity, and threw new light on nearly all parts of pure or abstract mathematics. In 1735, an astronomical problem proposed by the Academy, for the solution of which several eminent mathematicians had demanded several months' time, was solved by Euler in three days with the aid of improved methods of his own, but the effort threw him into a fever which endangered his life and deprived him of his right eye, his eyesight having been impaired by the severity of the climate. With still superior methods, this same problem was solved later by the illustrious German mathematician, Gauss.

In 1741, at the request, or rather command, of Frederick the Great, he moved to Berlin, where he was made a member of the Academy of Sciences, and Professor of Mathematics. He enriched the last volume of the *Mélanges* or Miscellanies of Berlin, with five memoirs, and these were followed, with astonishing rapidity, by a great number of important researches, which were scattered throughout the annual memoirs of the Prussian Academy. At the same time, he continued his philosophical contributions to the Academy of St. Petersburg, which granted him a pension in 1742.

The respect in which he was held by the Russians was strikingly shown in 1760, when a farm he occupied near Charlottenburg happened to be pillaged by the invading Russian army. On its being ascertained that the farm belonged to Euler, the general immediately ordered compensation to be paid, and the Empress Elizabeth sent an additional sum of four thousand crowns. The despotism of Anne I. caused Euler, who was a very timid man, to shrink from public

affairs, and to devote all his time to science. After his call to Berlin, the Queen of Prussia who received him kindly, wondered how so distinguished a scholar should be so timid and reticent. Euler replied, "Madam, it is because I come from a country where, when one speaks, one is hanged."

In 1766, Euler, with difficulty, obtained permission from the King of Prussia to return to St. Petersburg, to which he had been originally called by Catherine II. Soon after returning to St. Petersburg a cataract formed in his left eye, which ultimately deprived him of sight, but this did not stop his wonderful literary productiveness, which continued for seventeen years—until the day of his death. It was under these circumstances that he dictated to his amanuensis, a tailor's apprentice who was absolutely devoid of mathematical knowledge, his *Anleitung zur Algebra*, or *Elements of Algebra*, 1770, a work which, though purely elementary, displays the mathematical genius of its author, and is still considered one of the best works of its class. Euler was one of the very few great mathematicians who did not deem it beneath the dignity of genius to give some attention to the recasting of elementary processes and the perfecting of elementary text-books, and it is not improbable that modern mathematics is as greatly indebted to him for his work along this line as for his original creative work.

Another task to which he set himself soon after returning to St. Petersburg was the preparation of his *Lettres à une Princesse d'Allemagne sur quelques sujects de Physique*, (3 vols. 1768-72). These letters were written at the request of the princess of Anhalt-Dessau, and contain an admirably clear exposition of the principal facts of mechanics, optics, acoustics, and physical astronomy. Theory, however, is frequently unsoundly applied in it, and it is to be observed generally that Euler's strength lay rather in pure than in applied mathematics. In 1755, Euler had been elected a foreign member of the Academy of Sciences at Paris, and sometime afterwards the academical prize was adjudged to three of his memoirs *Concerning the Inequalities in the Motions of the Planets*. The two prize-problems proposed by the same Academy in 1770 and 1772 were designed to obtain a more perfect theory of the moon's motion. Euler, assisted by his eldest son, Johann Albert, was a competitor for these prizes and obtained both. In his second memoir, he reserved for further consideration the several inequalities of the moon's motion, which he could not determine in his first theory on account of the complicated calculations in which the method he then employed had engaged him. He afterward reviewed his whole theory with the assistance of his son and Krafft and Lexell, and pursued his researches until he had constructed the new tables, which appeared with the great work in 1772. Instead of confining himself, as before, to the fruitless integration of three differential equations of the second degree, which are furnished by mathematical principles, he reduced them to three ordinates which determine the place of the moon ; and he divides into classes all the inequalities of that planet, as far as they depend either on the elongation of the sun and moon, or upon the eccentricity, or the parallax, or the inclination of the lunar orbit. The inherent difficulties of this

task were immensely enhanced by the fact that Euler was virtually blind, and had to carry all the elaborate computations involved in his memory. A further difficulty arose from the burning of his house and the destruction of a greater part of his property in 1771. His manuscripts were fortunately preserved. His own life only was saved by the courage of a native of Basel, Peter Grimmon, who carried him out of the burning house.

Some time after this, the celebrated Wenzell, by couching the cataract, restored his sight ; but a too harsh use of the recovered faculty, together with some carelessness on the part the surgeons, brought about a relapse. With the assistance of his sons, and of Krafft and Lexell, however, he continued his labors, neither the loss of his sight nor the infirmities of an advanced age being sufficient to check his activity. Having engaged to furnish the Academy of St. Petersburg with as many memoirs as would be sufficient to complete its acts for twenty years after his death, he in seven years transmitted to the Academy above seventy memoirs, and left above two hundred more, which were revised and completed by another hand.

Euler's knowledge was more general than might have been expected in one who had pursued with such unremitting ardor, mathematics and astronomy, as his favorite studies. He had made considerable progress in medicine, botany, and chemistry, and he was an excellent classical scholar and extensively read in general literature. He could repeat the *Æneid* of Virgil from the beginning to the end without hesitation, and indicate the first and last line of every page of the edition which he used. But such lines from Virgil as, "The anchor drops, the rushing keel is staid," always suggested to him a problem and he could not help enquiring what would be the ship's motion in such a case.

Euler's constitution was uncommonly vigorous and his general health was always good. He was enabled to continue his labors to the very close of his life so that it was said of him, that he ceased to calculate and to breath at nearly the same moment. His last subject of investigation was the motions of balloons, and the last subject on which he conversed was the newly discovered planet Herschel.

On the 18th of September, 1783, while he was amusing himself at tea with one of his grandchildren, he was struck with apoplexy, which terminated the illustrious career of this wonderful genius, at the age of seventy-six. His works, if printed in their completeness, would occupy from 60 to 80 quarto volumes. However, no complete edition of Euler's writings has been published, though the work has been begun twice.

He was simple, upright, affectionate, and had a strong religious faith. His single and unselfish devotion to the truth and his joy at the discoveries of science whether made by himself or others, were striking attributes of his character. He was twice married, his second wife being a half-sister of his first, and he had a numerous family, several of whom attained to distinction. His *éloge* was written for the French Academy by Condorcet, and an account of his life, with a list of his works, was written by Von Fuss, the secretary of the Imperial Academy of St. Petersburg.

As has been said, Euler wrote an immense number of works, chief of which are the following: *Introductio in Analysin infinitorum*, 1748, which was intended to serve as an introduction to pure analytical mathematics. This work produced a revolution in analytical mathematics, as the subject of which it treated had hitherto never been presented in so general and systematic a manner. The first part of the Analysis Infinitorum contains the bulk of the matter which is to be found in modern text-books on algebra, theory of equations, and trigonometry. In the algebra, he paid particular attention to the expansion of various functions in series, and to the summation of given series; and pointed out explicitly that an infinite series can not be safely employed in mathematical investigations unless it is convergent. In trigonometry, he introduced (simultaneously with Thomas Simpson in England) the now current abbreviations for trigonometric functions, and simplified formulæ by the simple expedient of designating the angles of a triangle by A, B, C, and the opposite sides by a, b, c. He also showed that the trigonometrical and exponential functions are connected by the the relation $\cos\theta + i\sin\theta = e^{i\theta}$. Here too we meet the symbol e used to denote the base of the Naperian logarithms, namely the incommensurable number 2.7182818 . . . and the symbol π used to denote the incommensurable number 3.14159265 . . . The use of a single symbol to denote the number 2.7182818 . . . seemes to be due to Cotes, who denoted it by M. Newton was probably the first to employ the literal exponential notation, and Euler using the form a^z, had taken a as the base of any system of logarithms. It is probable that the choice of e for a particular base was determined by its being the vowel consecutive to a, or, still more probable because e is the initial of the word *exponent*.

The use of a single symbol to denote 3.14159265 . . . appears to have been introduced by John Bournilli, who represented it by c. Euler in 1734 denoted it by p, and in a letter of 1736 in which he enunciated the theorem that the sum of the square of the reciprocals of the natural numbers is $\frac{1}{6}\pi^2$, he uses the letter c. Chr. Goldbach in 1742 used π, and after the publication of Euler's Analysis, the sympol π was generally employed, the choice of π being determined by the initial of the word, $\pi\varepsilon\rho\iota\phi\acute{\varepsilon}\rho\varepsilon\iota\alpha$=*periphereia*.

The second part of the *Analysis Infinitorum* is on analytical geometry. Euler begins this part by dividing curves into algebraic and transcendental, and establishes a number of propositions which are true for all algebraic curves. He then applied these to the general equation of the second degree in two dimensions, showed that it represents the various conic sections, and deduces most of their properties from the general equation. He also considered the classification of cubic, quartic, and other algebraic curves. He next discussed the question as to what surfaces are represented by the general equation of the second degree in three dimensions, and how they may be discriminated one from the other. Some of these surfaces had not been previously investigated. In this work he also laid down the rules for the transformation of coördinates in space. Here also we find the first attempt to bring the curvature of surfaces within the domain of mathematics, and the first complete discussion of tortuous curves.

In 1755 appeared *Institutiones Calculi Differentialis*, to which the *Analysis Infinitorum* was intended as an introduction. This is the first text-book on the differential calculus which has any claim to be regarded as complete, and it may be said that most modern treatises on the subject are based upon it.

At the same time, the exposition of the principles of the subject is often prolix and obscure, and sometimes not quite accurate.

This series of works was completed by the publication in three volumes in 1768 to 1770 of the *Institutiones Calculi Integralis*, in which the results of several of Euler's earlier memoirs on the same subjects and on differential equations are included. In this treatise as in the one on the differential calculus was summed up all that was at that time known on the subject. The beta and gamma functions were invented by Euler, and are discussed here, but only as methods of reduction and integration. His treatment of elliptic integrals is superficial. The classic problems on isoperimetrical curves, the brachistochrone in a resisting medium, and theory of geodesics had engaged Euler's attention at an early date, and the solving of which led him to the calculus of variations. The general idea of this was laid down in his *Curvarum Maximi Minimive Proprietate Gaudentium Inventio Nova ac Facilis*, published in 1744, but the complete development of the new calculus was first effected by Lagrange in 1759. The method used by Lagrange is described in Euler's integral calculus, and is the same as that given in most modern text-books on the subject.

In 1770, Euler published the *Anleitung zur Algebra* in two volumes. The first volume treats of determinate algebra. This work includes the proof of the binomial theorem for any index, which is still known by Euler's name. The proof, which is not accurate according to the modern views of infinite series, depends upon the principle of the permanence of equivalent forms, and may be seen in C. Smith's *Treatise on Algebra*, pages 336-7. Euler's proof with important additions due to Cauchy, may be seen in G. Chrystal's Algebra, Part II.

It is a fact worthy of note that Euler made no attempt to investigate the convergency of the series, though he clearly recognized the necessity of considering the convergency of infinite series. While Euler recognized the convergency of series, his conclusions in reference to infinite series are not always sound. In his time no clear notion as to what constitutes a convergent series existed, and the rigid treatment to which infinite series are now subjected was undreamed of. Euler concluded that the sum of the oscillating series $1-1+1-1+1-1+\ldots = \frac{1}{2}$, for the reason, that by stopping with an even number of terms the sum is 0, and by stopping with an odd number of terms the sum is 1. Hence, the sum of the series is $\frac{1}{2}(0+1)=\frac{1}{2}$. Guido Grandi went so far as to conclude that $\frac{1}{2}=0+0+0+0\ldots$ The paper in which Euler cautions against divergent series contains the proof that $\ldots \frac{1}{n^2}+\frac{1}{n}+1+n+n^2+n^3\ldots = 0$. His proof is as follows, $n+n^2+n^3+\ldots = \frac{n}{1-n}$, $1+\frac{1}{n}+\frac{1}{n^2}+\ldots = \frac{n}{n-1}$, $\frac{n}{n-1}+\frac{n}{1-n} = 0$. Euler had no hesitation in writing $1-3+5-7+9-\ldots = 0$, and he confidently believed that $\sin\phi-2\sin2\phi+3\sin3\phi-\ldots = 0$.

A remarkable development, due to Euler, is what he named the hypergeometrical series, the summation of which he observed to be dependent upon the integration of linear differential equations of the second order, but it remained for Gauss to point out that for special values of the letters, this series represented nearly all the functions then known. By giving the factors 641×6700417 of the number $2^{2^n}+1=4294967297$ when $n=5$, he pointed out the fact that this expression did not always represent primes, as was supposed by Fermat.

The sources from which this biography has been obtained are Cajori's and Ball's *History of Mathematics*, and the *Encyclopedia Britannica*.

MOMENTS OF INERTIA.

By G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

It is the purpose of this paper to put on record formulæ for the Moments of Inertia of the plane areas, $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}=1$, and the solid bounded by the surface, $\left(\frac{x}{a}\right)^{\frac{2}{2m+1}}+\left(\frac{y}{b}\right)^{\frac{2}{2n+1}}+\left(\frac{z}{c}\right)^{\frac{2}{2p+1}}=1$.

Let μ be the mass of a unit, (a) area, (b) volume.

(a) Areas, when n and m are positive integers.

For the x-axis,

$$I=4\mu\int\int y^2dxdy=\frac{4ab^3\mu}{4}\cdot\frac{\Gamma(m+\frac{1}{2})\Gamma(3n+\frac{3}{2})}{(2m+1)(2n+1)\Gamma(m+3n+3)}$$

$$=\frac{\mu ab^3(2m+1)(2n+1)(6n+1)\Gamma(m+\frac{1}{2})\Gamma(3n+\frac{1}{2})}{2(m+3n+2)(m+3n)(m+3n+1)\Gamma(m+3n)}\ldots\ldots\ldots\ldots(1)$$

$$=\frac{1.3.5.\ldots\ldots(2m+1)\times 1.3.5\ldots\ldots(6n+1)}{2\ 4.6.\ldots\ldots 2(m+3n+2)}\cdot 2\pi\mu ab^3(2n+1)\ldots\ldots\ (2).$$

For the y-axis,

$$I_1=4\mu\int\int x^2dxdy=\frac{\mu a^3b(2m+1)(2n+1)(6m+1)\Gamma(3m+\frac{1}{2})\Gamma(n+\frac{1}{2})}{2(3m+n+2)(3m+n+1)(3m+n)\Gamma(3m+n)}\ldots\ldots\ldots(3)$$

$$=\frac{1.3.5.\ldots\ldots(6m+1)\times 1.3.5\ldots\ldots(2n+1)}{2.4.6.\ldots\ldots 2(3m+n+2)}\cdot 2\pi\mu a^3b(2m+1)\ldots\ldots\ (4).$$

For an axis through its center perpendicular to its plane,

$$I_2=I+I_1\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5).$$

The product of inertia of a quadrant about its axes is,

$$p=\mu\int\int xydxdy=\frac{\mu a^2b^2}{4}\cdot\frac{\Gamma(2m+2)\Gamma(2n+1)}{(2m+1)(2n+1)\Gamma(2m+2n+3)}$$

$$=\frac{\mu a^2b^2mn(2m+1)(2n+1)\Gamma(2m)\Gamma(2n)}{4(m+n+1)(m+n)(2m+2n+1)\Gamma(2m+2n)}\ldots\ldots\ldots\ldots\ldots(6)$$

$$= \frac{1.2.3.4.\ldots\ldots(2m+1)\times 1.2.3.4.\ldots\ldots(2n+1)}{1.2.3.4.\ldots\ldots(2m+2n+2)} \cdot \frac{\mu a^2 b^2}{4} \quad \ldots\ldots\ldots(7).$$

Let $m=n=0$. Then for the ellipse, $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$.

$$I=\tfrac{1}{4}\pi\mu ab^3, \quad I_1=\tfrac{1}{4}\pi\mu a^3 b, \quad I_2=\tfrac{1}{4}\pi\mu ab(a^2+b^2), \quad p=\tfrac{1}{8}\mu a^2 b^2.$$

Let $m=n=1$. Then for the hypocycloid, $\left(\frac{x}{a}\right)^{\frac{2}{3}}+\left(\frac{y}{b}\right)^{\frac{2}{3}}=1$.

$$I=\tfrac{2}{5}\tfrac{1}{2}\pi\mu ab^3, \quad I_1=\tfrac{2}{5}\tfrac{1}{2}\pi\mu a^3 b, \quad I_2=\tfrac{2}{5}\tfrac{1}{2}\pi\mu ab(a^2+b^2), \quad p=\tfrac{1}{8}\tfrac{1}{0}\mu a^2 b^2.$$

(b) Solids, when m, n, and p are positive integers.
With regard to the plane (yz),

$$\mathbf{I}=8\mu\int\int\int x^2\,dxdydz$$

$$= \frac{8\mu a^3 bc}{\frac{8}{(2m+1)(2n+1)(2p+1)}} \cdot \frac{\Gamma(3m+\frac{3}{2})\Gamma(n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{\Gamma(3m+n+p+\frac{7}{2})}$$

$$= \frac{4\mu a^3 bc(2m+1)(2n+1)(2p+1)(6m+1)\Gamma(3m+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{(6m+2n+2p+5)(6m+2n+2p+3)(6m+2n+2p+1)\Gamma(3m+n+p+\frac{1}{2})} \quad \ldots\ldots(8)$$

$$= \frac{1.3.5.\ldots\ldots(6m+1)\times 1.3.5.\ldots\ldots(2n+1)\times 1.3.5.\ldots\ldots(2p+1)}{1.3.5.\ldots\ldots(6m+2n+2p+5)}$$

$$\times 4\mu\pi a^3 bc(2m+1).\ldots\ldots(9).$$

With regard to the plane (xz),

$$\mathbf{I}_1=8\mu\int\int\int y^2\,dxdydz$$

$$= \frac{4\mu ab^3 c(2m+1)(2n+1)(2p+1)(6n+1)\Gamma(m+\frac{1}{2})\Gamma(3n+\frac{1}{2})\Gamma(p+\frac{1}{2})}{(2m+6n+2p+5)(2m+6n+2p+3)(2m+6n+2p+1)\Gamma(m+3n+p+\frac{1}{2})} \quad \ldots(10)$$

$$= \frac{1.3.5.\ldots\ldots(2m+1)\times 1.3.5.\ldots\ldots(6n+1)\times 1.3.5.\ldots\ldots(2p+1)}{1.3.5.\ldots\ldots(2m+6n+2p+5)}$$

$$\times 4\mu\pi ab^3 c(2n+1).\ldots\ldots(11).$$

With regard to the plane (xy),

$$\mathbf{I}_2=8\mu\int\int\int z^2\,dxdydz$$

$$= \frac{4\mu abc^3(2m+1)(2n+1)(2p+1)(6p+1)\Gamma(m+\frac{1}{2})\Gamma(n+\frac{1}{2})\Gamma(3p+\frac{1}{2})}{(2m+2n+6p+5)(2m+2n+6p+3)(2m+2n+6p+1)\Gamma(m+n+3p+\frac{1}{2})} \quad \ldots(12)$$

$$= \frac{1.3.5.\ldots\ldots(2m+1)\times 1.3.5.\ldots\ldots(2n+1)\times 1.3.5.\ldots\ldots(6p+1)}{1\,3.5.\ldots\ldots(2m+2n+6p+5)}$$

$$\times 4\mu\pi abc^3(2p+1)\ldots\ldots(13).$$

$\mathbf{I}_3=\mathbf{I}+\mathbf{I}_1$, for z-axis, $\mathbf{I}_4=\mathbf{I}+\mathbf{I}_2$, for y-axis,

$\mathbf{I}_5=\mathbf{I}_1+\mathbf{I}_2$, for x-axis, $\mathbf{I}_6=\mathbf{I}+\mathbf{I}_1+\mathbf{I}_2$, for center.

Product of inertia of an octant of the solid with regard to the (y, z) axes,

$$P=\mu\int\int\int yzdxdydz=\frac{\mu ab^2c^2}{8}\cdot\frac{\Gamma(m+\frac{1}{2})\Gamma(2n+1)\Gamma(2p+1)}{\Gamma(m+2n+2p+\frac{7}{2})}$$

$$=\frac{4\mu ab^2c^2np(2m+1)(2n+1)(2p+1)\Gamma(m+\frac{1}{2})\Gamma(2n)\Gamma(2p)}{(2m+4n+4p+5)(2m+4n+4p+3)(2m+4n+4p+1)\Gamma(m+2n+2p+\frac{1}{2})}\quad\ldots(14)$$

$$=\frac{1.2.3.\ldots\ldots(2n+1)\times 1.2.3.\ldots\ldots(2p+1)\times\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.\ldots\ldots\left(\frac{2m+1}{2}\right)}{\frac{1}{2}.\frac{3}{2}.\frac{5}{2}.\frac{7}{2}\ldots\ldots\left(\frac{2m+4n+4p+5}{2}\right)}\cdot\frac{\mu ab^2c^2}{4}$$

$$=\frac{1.2.3.4.\ldots\ldots(2n+1)\times 1.2.3.4.\ldots\ldots(2p+1)}{(2m+3)(2m+5)\ldots\ldots(2m+4n+4p+5)}\cdot\mu ab^2c^2.2^{2(n+p)}\ldots(15).$$

With regard to the axes (x, z),

$$P_1=\mu\int\int\int xzdxdydz$$

$$=\frac{4\mu a^2bc^2mp(2m+1)(2n+1)(2p+1)\Gamma(2m)\Gamma(n+\frac{1}{2})\Gamma(2p)}{(4m+2n+4p+5)(4m+2n+4p+3)(4m+2n+4p+1)\Gamma(2m+n+2p+\frac{1}{2})}\quad\ldots(16)$$

$$=\frac{2^{2(m+p)}1.2.3.4\ldots\ldots(2m+1)\times 1.2.3\,4.\ldots\ldots(2p+1)}{(2n+3)(2n+5)\ldots\ldots(4m+2n+4p+5)}\mu a^2bc^2\ldots\quad(17).$$

With regard to the axes (x, y),

$$P_2=\mu\int\int\int xydxdydz$$

$$=\frac{4\mu a^2b^2cmn(2m+1)(2n+1)(2p+1)\Gamma(2m)\Gamma(2n)\Gamma(p+\frac{1}{2})}{(4m+4n+2p+5)(4m+4n+2p+3)(4m+4n+2p+1)\Gamma(2m+2n+p+\frac{1}{2})}\quad\ldots(18)$$

$$=\frac{2^{2(m+n)}1.2.3.4.\ldots\ldots(2m+1)\times 1.2.3.4.\ldots\ldots(2n+1)}{(2p+3)(2p+5)\ldots\ldots(4m+4n+2p+5)}\cdot\mu a^2b^2c\ldots(19).$$

Let $m=n=p=0$. Then for $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2=1$,

$\mathbf{I}=\frac{4}{15}\mu\pi a^3bc$, $\mathbf{I}_1=\frac{4}{15}\mu\pi ab^3c$, $\mathbf{I}_2=\frac{4}{15}\mu\pi abc^3$,

$\mathbf{I}_3=\frac{4}{15}\mu\pi abc(a^2+b^2)$, $\mathbf{I}_4=\frac{4}{15}\mu\pi abc(a^2+c^2)$,

$\mathbf{I}_5=\frac{4}{15}\mu\pi abc(b^2+c^2)$, $\mathbf{I}_6=\frac{4}{15}\mu\pi abc(a^2+b^2+c^2)$,

$P=\frac{1}{15}\mu ab^2c^2$, $P_1=\frac{1}{15}\mu a^2bc^2$, $P_2=\frac{1}{15}\mu a^2b^2c$.

Let $m=n=p=1$. Then for $\left(\frac{x}{a}\right)^3+\left(\frac{y}{b}\right)^3+\left(\frac{z}{c}\right)^3=1$,

$\mathbf{I}=\frac{4}{105}\mu\pi a^3bc$, $\mathbf{I}_1=\frac{4}{105}\mu\pi ab^3c$, $\mathbf{I}_2=\frac{4}{105}\mu\pi abc^3$,

$\mathbf{I}_3=\frac{4}{105}\mu\pi abc(a^2+b^2)$, $\mathbf{I}_4=\frac{4}{105}\mu\pi abc(a^2+c^2)$,

$\mathbf{I}_5=\frac{4}{105}\mu\pi abc(b^2+c^2)$, $\mathbf{I}_6=\frac{4}{105}\mu\pi abc(a^2+b^2+c^2)$,

$P=\frac{64\mu ab^2c^2}{15.13.11.7.5}$, $P_1=\frac{64\mu a^2bc^2}{15.13.11.7.5}$, $P_2=\frac{64\mu a^2b^2c}{15.13.11.7.5}$.

Thus we could multiply examples without number.

Formulæ (1), (3), (6), (8), (10), (12), (14), (16), (18), will hold for m, n, p fractional as well as integral.

For the radius of gyration we have

$$k_n^2=\frac{I_n}{\mu A},\quad K_n^2=\frac{\mathbf{I}_n}{\mu V},$$

where A and V are known, (see AMERICAN MATHEMATICAL MONTHLY, page 380, Vol. I., No. 11.)

A SIMPLE DEDUCTION OF THE DIFFERENTIAL OF LOGx.

By J. W. NICHOLSON. A. M., LL. D., Professor of Mathematics in Louisiana State University.

Let $f(x)=\log x$ ..... (1), then $f(xy)=f(x)+f(y)$ ..... (2).

Differentiate, $f'(xy)(ydx+xdy)=f'(x)dx+f'(y)dy$ ..... (3).

Since (3) is true when x and y are independent,

$f'(xy)ydx=f'(x)dx$ ..... (4), and $f'(xy)xdy=f'(y)dy$ ..... (5).

$$(4)\div(5),\quad \frac{f'(x)}{f'(y)}=\frac{y}{x}=\frac{1/x}{1/y} \quad \ldots\ldots (6).$$

$$\therefore f'(x)=\frac{m}{x},\quad f'(y)=\frac{m}{y} \quad \ldots\ldots (7). \quad \therefore d\log x=\frac{m}{x}dx \quad \ldots\ldots (8).$$

NON-EUCLIDEAN GEOMETRY: HISTORICAL AND EXPOSITORY.

By GEORGE BRUCE HALSTED, A. M. (Princeton); Ph. D. (Johns Hopkins); Member of the London Mathematical Society; and Professor of Mathematics in the University of Texas, Austin, Texas.

[Continued from November Number.]

PROPOSITION XXX. *To any terminated straight AB stands at right angles* (*Fig.* 36.) *a certain unbounded straight BX. I say firstly, that the straight AY, erected perpendicularly toward the same parts upon AB, will be one intrinsic limit of all those straights, which drawn from the point A out toward the same parts have* (*in hypothesis of acute angle*) *a common perpendicular in two distinct points with the other unbounded straight BX. I say secondly that no acute angle will be the minimum of all, produced under which a straight from the aforesaid point A* (*in the aforesaid hypothesis*) *has in two distinct points a common perpendicular with BX.*

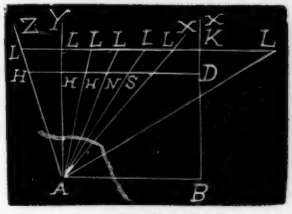


Fig. 36.

Proof of the first part.

For since AY has in common at two distinct points A and B the perpendicular AB with BX; if any straight AZ is drawn toward the same parts under an obtuse angle, it follows there can be toward these parts in two distinct points no common perpendicular to AZ, BX. Otherwise from the resulting quadrilateral containing four angles greater than four right angles, we hit (from Proposition XVI.) upon the already rejected hypothesis of obtuse angle, against the hypothesis of acute angle in this place assumed.

Therefore that perpendicular AY will be from that side an intrinsic limit of all the straights which drawn from the point A toward the same parts have (in the hypothesis of acute angle) at two distinct points a common perpendicular with the other unbounded straight BX. Quod erat primum.

Proof of the second part.

For if it were possible, let a certain acute angle be the least of all, drawn under which AN has with BX in two distinct points the common perpendicular ND. Then in BX a higher point K being assumed, from this erect to BX the perpendicular KL, upon which from the point A let fall (by Euclid I. 12) the perpendicular AL.

But now, if this AL meets ND in any point S, it certainly follows that angle BAL will be less than BAN, which therefore will not be the least of all drawn under which AN has with BX in two distinct points a common perpendicular ND.

But furthermore that the aforesaid perpendicular ND is cut by this perpendicular AL in some intermediate point of it S is thus demonstrated.

And first indeed, that BK cannot be cut by AL in any point M follows absolutely from Euclid I. 17, since otherwise in the same triangle MKL we would have two right angles at the points K and L, apart from the fact that in this case we would have our assertion about that angle BAN, that it is not in such circumstances the least of all.

But again AL cannot be the continuation of AN; because otherwise in the quadrilateral $NDKL$ we would have four right angles, against the hypothesis of acute angle.

But neither can it cut DN produced in any exterior point H; because angle AHN (from Euclid I. 16) would be acute, on account of the external angle AND supposed right; and therefore angle DHL would be obtuse, and so in the quadrilateral $DHLK$ we would have four angles, which taken together would be greater than four right angles, against the aforesaid hypothesis of acute angle.

Therefore it follows that the angle BAN must be cut by this AL, and therefore cannot be declared the least of all, drawn under which AN has with BX in two distinct points a common perpendicular ND.

Quod erat secundo loco demonstrandum. Itaque constat etc.

COROLLARY. But hence is permitted to observe, that under a lesser angle BAL is obtained (in hypothesis of acute angle) a common perpendicular LK, more remote indeed from the base AB, as follows from the construction, but moreover less than the other nearer common perpendicular ND, which is obtained under a greater angle BAN.

The reason of this latter is because in the quadrilateral $LKDS$ the angle at the point S is acute in the aforesaid hypothesis, since the three remaining angles are supposed right.

Wherefore (from Corollary I. to Proposition III.) the side LK will be less than the opposite side SD, and so much less than the side ND.

[To be Continued.]

SOPHUS LIE'S TRANSFORMATION GROUPS.

A SERIES OF ELEMENTARY, EXPOSITORY ARTICLES.

By EDGAR ODELL LOVETT, Princeton University.

III.

CONSTRUCTION OF A ONE PARAMETER GROUP FROM AN INFINITESIMAL TRANSFORMATION.

9. Let there be given the one parameter continuous group

$$x_1 = \varphi(x, y, a), \quad y_1 = \psi(x, y, a); \tag{1}$$

assume further that it contains the inverse transformation of every transformation in it, *i. e.* that the solutions of the equations (1) with regard to x and y have the form

$$x=\varphi(x_1, y_1, b), \quad y=\psi(x_1, y_1, b),$$

in which b is a canstant depending only on a. In the preceding paragraphs the theorem of LIE that every one parameter group whose transformations are inverse in pairs contains an infinitesimal transformation was arrived at both geometrically and analytically. Either process may be formulated symbolically as follows. If T_a represent the transformation of the group corresponding to the parameter a, its inverse T_a^{-1} is also contained in (1) by hypothesis. Further $T_{a+\delta a}$ will represent the transformation corresponding to the parameter $a+\delta a$, and therefore the transformation of the group (1) that differs from T_a by an infinitesimal. The successive application or the product of $T_{a+\delta a}$ and T_a^{-1}, namely $T_{a+\delta a}T_a^{-1}$ (which belongs to the group by virtue of our first supposition that the product of any two transformations of the group is itself a transformation of the group), differs infinitesimally from $T_aT_a^{-1}$, the identical transformation, and hence is itself an infinitesimal transformation belonging to the group (1).

10. On the other hand there is always a completely determinate continuous group of transformations which contains a given infinitesimal transformation. The truth of this assertion may be made to appear symbolically in the following manner.

Let S be any arbitrary transformation in the xy-plane. Construct the transformations which are equivalent to the repetition of S once, twice, and so on to n-times ; also the inverse of S, S^{-1}, and those equivalent to the repetition of this inverse once, twice, and so on to n-times ; we then have an infinite family of transformations,

$$\ldots, S^{-n}, \ldots, S^{-2}, S^{-1}, S^0, S^1, S^2, \ldots, S^n, \ldots,$$

where S^0 is the identical transformation, while n represents every possible positive whole number. This infinite family is a group, since if p and q are two positive or negative numbers, the product of S^p and S^q is equivalent to S^{p+q}, but the group is a discontinuous one.

In this manner, beginning with an arbitrary transformation S an infinite number of discontinuous groups in x and y may be constructed. Passing now to the limiting case, if, in particular, S is an infinitesimal transformation, then S^n and S^{n+1} differ from each other by an infinitesimal, and we have accordingly a continuous group constructed from, and containing the infinitesimal transformation, S.

11. LIE has invented an ingenious kinematical illustration of this limiting case, which serves as a concrete introduction to the rigorous demonstration of the theorem.

The infinitesimal transformation is defined by two equations of the form

$$x'=x+\xi(x, y)\delta t+\ldots, \quad y'=y+\eta(x, y)\delta t\ldots, \qquad (2)$$

where ξ and η are any two given functions of x and y, the quantity δt an infinitesimal, and the terms omitted convergent power series in δt beginning with δt^2.

The coördinates of the transformed point (x', y') differ from those of the original point (x, y) by the infinitesimal increments

$$\delta x = \xi(x, y)\delta t, \quad \delta y = \eta(x, y)\delta t,$$

when terms of the second order of infinitesimals are neglected. The infinitesimal transformation makes correspond to every point (x, y) an infinitesimal arrow (say) whose length is

$$\sqrt{\delta x^2 + \delta y^2} = \sqrt{\xi^2 + \eta^2}\,\delta t,$$

and direction

$$\frac{\delta y}{\delta x} = \frac{\eta}{\xi};$$

and in general to different points arrows of different lengths and different directions. The infinitesimal transformation thus puts all the points (x, y) of the plane in motion, and if the variable t be taken as the time, these points describe in the element of time δt, the infinitesimal paths $\sqrt{\xi^2 + \eta^2}\,\delta t$, whose projections on the axes are $\xi\delta t$ and $\eta\delta t$. In the first element of time δt the point (x, y) goes over into (x', y') describing the path $\sqrt{\xi(x, y)^2 + \eta(x, y)^2}\,\delta t$, in the next element δt it runs over the infinitesimal path $\sqrt{\xi(x', y')^2 + \eta(x', y')^2}\,\delta t$, and so on. The original point (x, y) assumes, by the continued application of the infinitesimal transformation, a continuous series of positions which may be represented by a curve. This motion of the points of the plane is characterized by the fact that the components of the velocity of every point (x, y) have the values

$$\frac{dx_1}{dt} = \xi(x_1, y_1), \quad \frac{dy_1}{dt} = \eta(x_1, y_1),$$

which depend only on the position and not on the time. Since the change of position is to repeat itself from moment to moment, the motion is a so-called stationary motion and can be compared to the flow of the particles of a compressible fluid. That the phenomena of a stationary motion exhibit the group property is readily seen, for if the stationary motion carries the points (x, y) to the position (x_1, y_1) in the time t_1 and then these new points (x_1, y_1) to the positions (x_2, y_2) in the time t_2, it is clear that the motion carries the original points (x, y) to the positions (x_2, y_2) in the time $t_1 + t_2$; *i. e.* the successive performance of two transformations (t_1) and (t_2) of the family is equivalent to a single transformation $(t_1 + t_2)$ of the family.

12. This kinematical illustration may now be replaced by the following rigorous analytical reasoning.

The two differential equations

$$\frac{dx_1}{dt} = \xi(x_1, y_1), \quad \frac{dy_1}{dt} = \eta(x_1, y_1), \tag{3}$$

determine x_1 and y_1 as functions of t, and the initial values corresponding to $t = 0$

which we take as $x_1=x$, $y_1=y$. In order to determine these functions x_1 and y_1, it is necessary to integrate the simultaneous system

$$\frac{dx_1}{\xi(x_1,y_1)}=\frac{dy_1}{\eta(x_1,y_1)}=dt, \tag{4}$$

with the initial conditions that $x_1=x$ and $y_1=y$ for $t=0$.

This integration is effected as follows. The differential equation in x_1, y_1

$$\frac{dx_1}{\xi(x_1,\,y_1)}=\frac{dy_1}{\eta(x_1,\,y_1)}$$

has an integral, $\Omega(x_1, y_1)$, which, since it is free from t, is also an integral of the whole simultaneous system (4). In order to find the second integral of the system which contains t, we eliminate say y_1 between the two equations

$$\Omega(x_1,\,y_1)=\text{constant}=c, \quad \text{and} \quad \frac{dx_1}{\xi(x_1,\,y_1)}=dt,$$

and obtain a differential equation,

$$\frac{dx_1}{\theta(x_1,\,c)}=dt.$$

Since the left hand member of this equation does not contain t it can be integrated by a quadrature* and its integral has the form $f(x_1, c)-t$. But this is not an integral of the system (4) until c has been eliminated by means of the equation $\Omega(x_1, y_1)=c$. Eliminating c, the second integral of the system (4) appears in the form $W(x_1, y_1)-t$.†

Finally, determining the constants of integration by the initial conditions that $x_1=x$, $y_1=y$ for $t=0$, we have as the result of the integration

$$\begin{aligned}\Omega(x_1,\,y_1)&=\Omega(x,\,y),\\ W(x_1,\,y_1)-t&=W(x,\,y).\end{aligned} \tag{5}$$

Without solving these equations for x_1, y_1 it is easy to see that they define a one parameter group, for the transformation of the family (5) which corresponds to the parameter value t carries the points (x, y) into the points (x_1, y_1), whose coördinates can be found by solving the equations (5) for x_1, y_1. A sec-

---

*By the term quadrature is meant an integral of the form $\int F(x)dx$. It is assumed that a quadrature can always be performed.

†The reader will observe that this same integral would have been found had we begun by eliminating x_1 from $\frac{dy_1}{\eta(x_1,\,y_1)}=dt$ by means of $\Omega(x_1, y_1)=c$. This elimination would have given the differential equation $\frac{dy_1}{\lambda(y_1,\,c)}=dt$; the integral of the latter, $\mu(y_1, c)-t$, is found by a quadrature; eliminating c by means of $\Omega(x_1, y_1)=c$, we have finally the second integral of the system, $W(x_1, y_1)-t$.

ond transformation of the same family with the parameter value t_1 will change the points (x_1, y_1) into the points (x_2, y_2) whose coördinates are found from the equations,

$$\Omega(x_2, y_2)=\Omega(x_1, y_1),$$
$$W(x_2, y_2)-t=W(x_1, y_1). \tag{6}$$

In order to find the transformation which carries the original points (x_1, y_1) directly into the final positions (x_2, y_2), it is only necessary to eliminate x_1, y_1 from the equations (5) and (6). The elimination gives at once

$$\Omega(x_2, y_2)=\Omega(x, y),$$
$$W(x_2, y_2)-(t+t_1)=W(x, y).$$

But these equations represent the transformation of the family (5) corresponding to the parameter value $t+t_1$; hence the family (5) possesses the group property. The group contains also the inverse transformation of every transformation in it and the identical transformation.

The equations (5) can be solved with regard to x_1, y_1 in the form

$$x_1=\Phi(x, y, t), \quad y_1=\Psi(x, y, t). \tag{7}$$

These two functions can be expanded in powers of t by Maclaurin's theorem. In order to effect the expansion we must have the values

$$\left(\frac{dx_1}{dt}\right)_{t=0}, \quad \left(\frac{d^2x_1}{dt^2}\right)_{t=0}, \ldots.$$

From equations (4) we have $\frac{dx_1}{dt}=\xi(x_1, y_1)$, with $x_1=x$, $y_1=y$, for $t=0$;

hence, $$\left(\frac{dx_1}{dt}\right)_{t=0}=\xi(x, y).$$

The equations (4) give also

$$\frac{d^2x_1}{dt^2}=\frac{\partial\xi(x_1, y_1)}{\partial x_1}\frac{dx_1}{dt}+\frac{\partial\xi(x_1, y_1)}{\partial y_1}\frac{dy_1}{dt}$$
$$=\frac{\partial\xi(x_1, y_1)}{\partial x_1}\xi(x_1, y_1)+\frac{\partial\xi(x_1, y_1)}{\partial y_1}\eta(x_1, y_1) ;$$

hence $$\left(\frac{d^2x_1}{dt^2}\right)_{t=0}=\frac{\partial\xi(x, y)}{\partial x}\xi(x, y)+\frac{\partial\xi(x, y)}{\partial y}\eta(x, y).$$

Similarly, $\left(\frac{dy_1}{dt}\right)_{t=0}=\eta(x, y)$, $\left(\frac{d^2y_1}{dt^2}\right)_{t=0}=\frac{\partial\eta(x, y)}{dx}\xi(x, y)+\frac{\partial\eta(x, y)}{dy}\eta(x, y).$

Accordingly equations (7) become by Maclaurin's theorem,

$$x_1 = x + \xi(x, y)t + \left(\xi\frac{\partial\xi}{\partial x} + \eta\frac{\partial\xi}{\partial y}\right)\frac{t^2}{1.2} + \ldots,$$

$$y_2 = y + \eta(x, y)t + \left(\xi\frac{\partial\eta}{\partial x} + \eta\frac{\partial\eta}{\partial y}\right)\frac{t^2}{1.2} + \ldots. \tag{8}$$

The reader will observe that $t=0$ in the equations (8) gives the identical transformation, and $t=\delta t$ gives an infinitesimal transformation which to terms of the second order agrees with the original infinitesimal transformation (2).

All these facts may now be summed up in the following theorem of LIE: *Every infinitesimal transformation*

$$x_1 = x + \xi(x, y)\delta t + \ldots, \quad y_1 = y + \eta(x, y)\delta t + \ldots,$$

belongs to at least one one parameter group with inverse transformations, when infinitesimals of the second and higher orders are neglected. The finite equations of this group are found by integrating the simultaneous system

$$\frac{dx_1}{\xi(x_1, y_1)} = \frac{dy_1}{\eta(x_1, y_1)} = dt,$$

with the initial conditions

$$x_1 = x, \quad y_1 = y, \quad \text{for } t=0,$$

in the form

$$\Omega(x_1, y_1) = \Omega(x, y),$$
$$W(x_1, y_1) - t = W(x, y);$$

or, solved with regard to x, y, and developed in powers of t, in the form

$$x_1 = x + \xi(x, y)\frac{t}{1!} + \left(\xi\frac{\partial\xi}{\partial x} + \eta\frac{\partial\xi}{\partial y}\right)\frac{t^2}{1!} + \ldots,$$

$$y_1 = y + \eta(x, y)\frac{t}{1!} + \left(\xi\frac{\partial\eta}{\partial x} + \eta\frac{\partial\eta}{\partial y}\right)\frac{t^2}{2!} + \ldots,$$

The one parameter group thus generated accordingly possesses an infinitesimal transformation which in its terms of the first order is identical with the original infinitesimal transformation.

We have now proved that every G_1 contains an infinitesimal transformation and conversely that every infinitesimal transformation generates a G_1. We shall prove in the next article that a G_1 contains but one infinitesimal transformation, with the converse that an infinitesimal transformation belongs to but one G_1. The theorems will be illustrated by concrete examples. These theorems establish the equivalence of the notions one parameter group and infinitesimal transformation; that these notions may be used interchangeably is the fundamental principle of LIE's Theory of the Group of One Parameter.

Princeton University, 14 December, 1897.

[To be Continued.]

ALGEBRA.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

74. Proposed by NELSON S. RORAY, South Jersey Institute, Bridgeton, N. J.

Solve according to the conditions given :

$$\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{1+x}}.$$

First, square without transposing and then solve ; second, transpose $\sqrt{x+1}$ and then solve. Obtain the same roots as in the first way of solving.

I. Solution by J. M. BOORMAN, Counselor, Inventor, etc., etc., Hewlett, L. I., N. Y.

Solve ("conditions given") $\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{1+x}}$ ..... (A).

The equation is of first degree. $\therefore$ *can have but one root, e. g.*

First. The *conditioned* operation gives, $2(x+1)+2\sqrt{(x+1)}\sqrt{x}=1+\frac{9}{1+x}$. Thence $\sqrt{x+1}.\sqrt{x}=-(x+1)+\frac{1}{2}+\frac{9}{2(1+x)}$.

Square, etc., and reduce : $\therefore 8\frac{3}{4}(1+x)^2-4\frac{1}{4}(1+x)=20\frac{1}{4}$.

Divide and supply : $\therefore (1+x)^2-\frac{1}{35}\frac{8}{5}(1+x)+\frac{8}{1}\frac{1}{2}\frac{9}{2}\frac{1}{5}=\frac{2}{1}\frac{9}{2}\frac{1}{2}\frac{6}{5}$.

$\therefore x=\frac{4}{5}$; $x_1=-\frac{1}{7}\frac{6}{7}$.

But, apply x_1 to the *given equation*. $\therefore \left(\frac{3+4}{\sqrt{7}}\right)\sqrt{-1}=\frac{3\sqrt{7}}{3\sqrt{-1}}=\frac{\sqrt{7}}{\sqrt{-1}}$.

Now $\frac{3+4}{\sqrt{7}}=\sqrt{7}$. $\therefore \sqrt{7}\sqrt{-1}=\sqrt{7}\left(\frac{1}{\sqrt{-1}}\right)$.

$\therefore -1\sqrt{7}=\sqrt{7}$; or $2\sqrt{7}=0$! ! So $x_1=-\frac{1}{7}\frac{6}{7}$ *is not a root* of equation (A), but of its factor $\sqrt{x+1}+\sqrt{x}=\frac{-3}{\sqrt{1+x}}$, that *inevitably* results by the conditioned involution. Hence $x=\frac{4}{5}$ *only*.

Second (direct) "way." Transpose and square.

$\therefore x=x+1-6+\frac{9}{1+x}$. Thence $x=\frac{4}{5}$, [the "*same root as in the first way.*"]

Proof. Apply *this* $x=\frac{4}{5}$, in equation (A).

$\therefore \sqrt{\frac{9}{5}}+\sqrt{\frac{4}{5}}=\frac{3\sqrt{5}}{\sqrt{9}}$. $\therefore$ as $\frac{3+2}{\sqrt{5}}=\frac{5}{\sqrt{5}}=\sqrt{5}$, so $\sqrt{5}=\frac{3\sqrt{5}}{\sqrt{9}}=\frac{3}{3}\sqrt{5}=1\sqrt{5}$,

i. e. $\sqrt{5}=\sqrt{5}$, *satisfies* equation (A). $\therefore x=\frac{4}{5}$ is the *one* root of (A). Q. E. D.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Squaring the equation as it stands, we get $2x+1+2\sqrt{x(x+1)}=\frac{9}{x+1}$.

Clearing of fractions and leaving the radical by itself in the first member, we get $2(x+1)\sqrt{x(x+1)}=8-3x-2x^2$. Squaring, arranging, and cancelling, we get the quadratic $35x^2+52x=64$, the two roots of which are $x=\frac{4}{5}$ and $-\frac{16}{7}$, the the former of which satisfies the equation $\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{x+1}}$, and the latter the equation $\sqrt{x+1}-\sqrt{x}=\frac{3}{\sqrt{x+1}}$.

Clearing the original equation of its denominator $\sqrt{x+1}$, we have $x+1+\sqrt{x(x+1)}=3$, or $\sqrt{x(x+1)}=2-x$. Squaring, we have $5x=4$. $\therefore x=\frac{4}{5}$.

III. Solution by F. M. McGAW, A. M., Professor of Mathematics in Bordentown Military Institute, Bordentown, N. J.; CHAS. C. CROSS, Laytonsville, Md.; G. B. M. ZERR, A. M., Ph. D., The Russell College, Lebanon, Va.; and J. P. BURDETTE, Class of '97, Dickinson College, Carlisle, Pa.

(1), $\sqrt{x+1}+\sqrt{x}=\frac{3}{\sqrt{1+x}}$. $2x+1+2\sqrt{x+x^2}=\frac{9}{1+x}$.

$\therefore 8-2x^2-3x=2(x+1)\sqrt{x+x^2}$. $\therefore 35x^2+52x=64$. $\therefore x=\frac{4}{5}$, or $-2\frac{2}{7}$.

(2). Regarding $\sqrt{x+1}$ as affected by the $\pm$ sign

$\sqrt{x}=\frac{2-x}{\sqrt{1+x}}$ or $\frac{4+x}{\sqrt{1+x}}$.

$\therefore x=(4-4x+x^2)/(1+x)$, or $(16+8x+x^2)/(1+x)$.

$\therefore x=\frac{4}{5}$, or $x=-2\frac{2}{7}$.

Also solved by *A. H. BELL*.

75. Proposed by the late B. F. BURLESON, Oneida Castle, N. Y.

Mr. B's farm is in shape a quadrilateral, both inscriptible and circumscriptible, and contains an area of $k=10752$ square rods. The square described on the radius of its inscribed circle contains $r^2=2304$ square rods; while the square described on the radius of its circumscribed circle contains an area of $R^2=7345$ square rods. Required the lengths of the sides of his farm.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

Let a, b, c, d be the sides required. By the conditions of the problem,

$a+c=b+d$; $abcd=k^2=115605504$ ..... (1).

$\frac{1}{2}r(a+b+c+d)=k$, or $a+b+c+d=2k/r=448$.

$\therefore a+c=b+d=k/r=224$ ..... (2).

$$R=\tfrac{1}{4}\sqrt{\frac{(ab+cd)(ac+bd)(bc+ad)}{abcd}}.$$

$\therefore (ab+cd)(ac+bd)(bc+ad)=16R^2k^2=13585958830080 \ldots\ldots\ldots\ldots (3).$

Substituting (2) in (1) and (3), we get

$$(224a-a^2)(224b-b^2)=115605504 \ldots\ldots\ldots\ldots\ldots\ldots (4).$$

$$\{ab+(224-a)(224-b)\}\{a(224-a)+b(224-b)\}\{b(224-a)+a(224-b)\}$$
$$=13585958830080 \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

Eliminating b from (4) and (5), we get, after reducing and factoring,

$$(a-168)(a-128)(a-96)(a-56)=0.$$

$\therefore$ The sides are 168, 128, 96, 56 rods, respectively.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.; and COOPER D. SCHMITT, A. M., Professor of Mathematics in the University of Tennessee, Knoxville, Tenn.

Denoting the four consecutive sides of the quadrilateral by a, b, c, d, we have, from well-known geometrical formulæ and principles:

$abcd=k^2 \ldots\ldots\ldots\ldots (1)$; $\quad r^2=abcd/[a+c]^2=k^2/[a+c]^2 \ldots\ldots\ldots\ldots (2)$;

$a+c=b+d=k/r \ldots\ldots (3)$; $\quad R^2=\{[ac=bd][ad+bc][ab+cd]\}/16k^2 \ldots\ldots (4).$

Putting $ac=x$, and $bd=y$, we have in (4),

$[x+y]\{ac[b^2+d^2]+y[a^2+c^2]\}=16k^2R^2$; or

$[x+y]\{x[(k^2/r^2)-2y]+y[(k^2/r^2)-2x]\}=16k^2R^2$; or

$[x+y]\{[x+y][k^2/r^2]-4xy\}=16k^2R^2$, and since $xy=abcd=k^2$,

$[x+y]\{[x+y]-4r^2\}=16R^2r^2$; or, reduced

$[x+y]^2-4r^2[x+y]=16R^2r^2$; whence $x+y=2r^2+2r\sqrt{r^2+4R^2}$,

and combining this with $xy=k^2$, we find x and y. Thus, we find for the given numerical values $x+y=21696$, $xy=115605504$, whence $x=12848$, $y=9408$. Now we have $ac=12288$, $a+c=224$, and $bd=9408$, $b+d=224$.

Whence $a=128$, $c=96$, $b=168$, $d=56$.

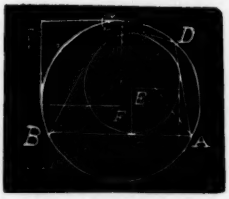
III. Solution by the PROPOSER.

Let $ABCD$ represent the farm, and let $x=CD$, $y=DA$, $z=AB$, $w=BC$, in order. We have $x+z=y+w \ldots\ldots\ldots\ldots (1)$. Also the following, where $s=x+y+z+w$, the perimeter of the quadrilateral:

$k^2=\frac{1}{16}\{[s-2x][s-2y][s-2z][s-2w]\} \ldots\ldots\ldots (2)$;

$rs=2k \ldots\ldots\ldots (3)$; $\quad k^2=xyzw \ldots\ldots\ldots (4)$;

$R^2=\{[xy+zw][xz+yw][xw+yz]\}$
$\div\{[s-2x][s-2y][s-2z][s-2w]\} \ldots (5).$



Substitute in (2), (4), and (5):

$m=xy+xz+xw+yz+yw+zw$;

$n=xyz+xyw+xzw+yzw$; and $p=xyzw$.

Then we shall have by involving terms and re-factoring,

$16k^2=16p-s^4+4s^2m-8sn$ .....(6) ; $k^2=p$. .....(7) ;

$R^2=16p-s^4+4s^2m-8sn=n^2-4pm+ps^2$ .....(8).

From (3), (6), (7), and (8), we obtain by elimination and resolution,

$s=2k/r=448$; $m=[k^3+r^3n]/r^2k=71872$; $n=2rk+2k\sqrt{[4R^2+r^2]}=4859904$;

$p=k^2=115605504$.

We now, by the "Theory of Equations," construct the biquadratic, the four roots of which will be the values of x, y, z, and w.

$x^4-448x^3+71872x^2-4859904x=-115605504$.....(9).

The four roots of equation (9), we find to be 56, 96, 128, and 168. Arranging these values in conformity with equation (1), we have, $CD=x=56$ rods, $DA=y=96$ rods, $AB=z=168$ rods, and $BC=w=128$ rods.

IV. Solution by A. H. BELL, Hillsboro, Illinois.

Since circumscriptible quadrilaterals have the sums of their opposite sides equal, take $x+y$, $x+z$, $x-y$, $x-z$, for the sides AB, BC, DC, and AD.

$\therefore 2rx=k$, $x=k/2r$ .....(1).

$\overline{BD}^2=[x+y]^2+[x-z]^2-2[x+y][x-z]\cos A$ .....(3).

$\overline{BD}^2=[x+z]^2+[x-y]^2+2[x-y][x+z]\cos A$,

$\{\cos C=\cos[180-A]=-\cos A\}$ .....(4).

$\therefore \cos A=\dfrac{x[y-z]}{x^2-y^2}$. $\sin^2 A=1-\cos^2 A=\dfrac{[x^2-y^2][x^2-z^2]}{[x^2-y^2]^2}$ .....(5).

$\overline{BD}^2=\dfrac{\{[x^2-y^2]+[x^2-z^2]\}[x^2+y^2]}{[x^2-y^2]}$; also $R^2=\dfrac{\overline{BD}^2}{4\sin^2 A}$ .....(6).

$2k=[x+y][x-z]\sin A+[x-y][x+z]\sin A$, or $[x^2-y^2]\sin A=k$.....(7).

Substituting the value of $\sin A$, (5) in (7), and

$[x^2-y^2][x^2-z^2]=k^2$ .....(8),

Then (6) becomes, $[x^2-y^2+x^2-z^2][x^4-y^2z^2]=4Rk^2$ .....(9).

Let the product of the opposite sides $v=x^2-y^2$. $\therefore y^2=x^2-v$....(10) ;

and $w=x^2-z^2$. $\therefore z^2=x^2-w$ .....(11).

Then (8), and (1), $vw=k^2=4r^2x^2$.....(12) ;

and (9) reduces to $[v+w]^2-4r^2[v+w]=16R^2r^2$ .....(13).

$\therefore\ v+w=2r^2\pm 2r\sqrt{4R^2+r^2}$ .....(14).

$(14)^2-4(7)$, etc. $v-w=2r[4R^2+2r^2-4x^2\pm 2r\sqrt{4R^2+r^2}]^{\frac{1}{2}}$ .....(15).

(14)$\pm$(15) after substituting the given values, $v=12288$, and $w=9408$.

(1), (5), and (6) $x=112$, $y=56$, and $z=16$, and the required sides AB, BC, CD, and AD are 168 rods, 128 rods, 56 rods, and 96 rods, respectively.

Also $BD=158.22$ rods.

Also solved by *CHARLES C. CROSS* and *H. C. WILKES*.

CALCULUS.

Conducted by J. M. COLAW, Monterey, Va. All contributions to this department should be sent to him.

SOLUTIONS OF PROBLEMS.

61. Proposed by W. H. CARTER, Professor of Mathematics, Centenary College of Louisiana, Jackson, La.

If $r=a\sin n\theta$ is the polar equation of a curve, show (1) that the curve consists of n or $2n$ loops according as n is an odd or an even integer; (2) that its area is $\frac{1}{4}$ or $\frac{1}{2}$ of the circumscribing circle according as n is an odd or an even integer.

I. Solution by G. B. M. ZERR, A. M., Ph. D., President and Professor of Mathematics in Russell College, Lebanon, Va.

$r=a\sin n\theta$. Let $r=0$, then $\sin n\theta=0$.

$\therefore\ \theta=0,\ 2\pi/n,\ 4\pi/n,\ 6\pi/n,\ 8\pi/n,\ \ldots\ldots$, are the angles at which the the curve cuts the polar axis at the pole.

$dr/db=na\cos n\theta=0$. $\therefore\ \theta=\pi/2n,\ 3\pi/2n,\ 5\pi/2n,\ 7\pi/2n,\ \ldots\ldots$, gives the points where r has its greatest value, namely, $\pm a$.

When n is odd the values of $n\theta$ for the angles $0,\ 2\pi/n,\ 4\pi/n,\ 6\pi/n,\ 8\pi/n,\ \ldots\ldots$, are $0,\ 2\pi,\ 4\pi,\ 6\pi,\ 8\pi,\ \ldots\ldots$

When n is even the values of $n\theta$ for the angles $0,\ 2\pi/n,\ 4\pi/n,\ 6\pi/n,\ 8\pi/n,\ \ldots\ldots$, are $0,\ \pi,\ 2\pi,\ 3\pi,\ 4\pi,\ \ldots\ldots$

$\therefore$ When n is even the polar axis is cut, at the pole, $2n$ times, but only n times when n is odd.

$A=$area of one loop.

$$A=\tfrac{1}{2}a^2\int_0^{\pi/n}\sin^2 n\theta\,d\theta,\ =\frac{\pi a^2}{4n}.$$

$$\therefore\ \frac{\pi a^2}{4n}\times n=\frac{\pi a^2}{4},\ \text{for } n \text{ odd};\ \frac{\pi a^2}{4n}\times 2n=\frac{\pi a^2}{2},\ \text{for } n \text{ even}.$$

II. Solution by E. L. SHERWOOD, A. M., Superintendent of City Schools, West Point, Miss.

Equation given $\rho = a\sin n\theta$. We may oberve that,

$$\rho = 0,\ a,\ 0,\ -a, \text{ etc., when}$$

$$\sin n\theta = 0,\ 1,\ 0,\ -1, \text{ etc., when}$$

$$n\theta = 0,\ \tfrac{1}{2}\pi,\ \pi,\ \tfrac{3}{2}\pi, \text{ etc., when}$$

$\theta = \{[c/n].\tfrac{1}{2}\pi\}$, where c is 0, 1, 2, 3, 4, etc., up to $4n$ ($4n$ being determined by $\theta = 2\pi$).

The series of values will be as follows:

$$\theta = 0.\frac{\pi}{2n},\quad \frac{\pi}{2n},\quad 2.\frac{\pi}{2n},\quad 3\frac{\pi}{2n} \ldots\ldots n.\frac{\pi}{2n},\quad [n+1]\frac{\pi}{2n} \ldots\ldots 2n\frac{\pi}{2n},$$

$$[2n+1]\frac{\pi}{2n} \ldots\ldots ;$$

If n is *even*, $\rho = 0,\ a,\ 0,\ -a \ldots\ldots 0 \pm a \ldots\ldots 0,\ a.$

If n is *odd*, $\rho = 0,\ a,\ 0,\ -a \ldots\ldots \pm a,\ 0 \ldots\ldots 0,\ -a \ldots\ldots$

$$\begin{cases} 3n.\dfrac{\pi}{2n}, & [3n+1]\dfrac{\pi}{2n} \ldots\ldots\ldots\ldots & 4n.\dfrac{\pi}{2n} \\ 0 & \pm a \ldots\ldots\ldots\ldots & 0 \\ \pm a & 0 \ldots\ldots\ldots\ldots & 0 \end{cases}$$

In each series are $4n$ terms (the first coincides with the last), and $\rho = a$ numerically in $2n$ of them. But when n is odd, the radius vector traces each loop twice for $\pi/2n$ and a is the same point as $\{[2n+1][\pi/2n]\}$ and $-a$.

$\therefore$ There are $2n$ loops when n is even, and n loops when n is odd.

$$\text{Area} = \tfrac{1}{2}\int \rho^2 d\theta, \text{ where } \rho^2 = a^2\sin^2 n\theta,$$

$$= \tfrac{1}{2}a^2 \int \sin^2 n d\theta,$$

$$= \tfrac{1}{2}a^2\left[\tfrac{1}{2}\theta - \frac{\sin 2n\theta}{4n}\right]_0^{\pi/2n} = \frac{\pi a^2}{8n} \text{ for } \tfrac{1}{2} \text{ loop},$$

or $\pi a^2/4n$ for an entire loop.

$\therefore$ For $2n$ loops, area $= \pi a^2/2$; and for n loops, area $= \pi a^2/4$.

Also solved by *J. SCHEFFER* and *C. W. M. BLACK*.

62. Proposed by A. H. HOLMES, Brunswick, Maine.

A bucket is in the form of a frustum of a cone having its smaller end as a base. It is a inches in diameter at base and b inches in diameter at top, and its perpendicular height is c inches. It contains water the perpendicular height of which is $\frac{1}{2}c$ inches. What is the greatest height, from the plane on which the vessel rests, to which the surface of the water will rise when the bucket is overturned, no allowance being made for the thickness of the material of the bucket?

Partial Solution by C. W. M. BLACK, A. M., Professor of Mathematics in Wesleyan Academy, Wilbraham, Mass.

Let $AEFL$ be section of frustum. Complete the cone to the apex N. Let NM be axis, HK surface of water, PQ plane on which vessel rests, AB height of surface of water above PQ. Denote $\angle ONA$ by α, ON by l, HK by x, HM by y.

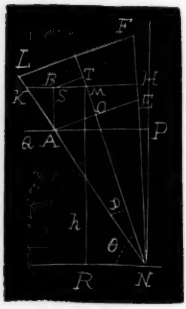


Fig. 1.

Then $EO=OA=\frac{1}{2}a$, $FL=b$. Denote angle which axis makes with PQ, $\angle ONR$ by θ, MR perpendicular to NR by h.

Now as vessel is tipped over, until H reaches E, volume of cone NHK is constant, and $=\frac{1}{3}\pi[\frac{1}{2}a+\frac{1}{2}b)^2[l+\frac{1}{2}c]$. Denote it by C.

Base HK is an ellipse, whose major axis is

$$x=h\cot[\theta-\alpha]-h\cot[\theta+\alpha] \quad \ldots\ldots\ldots\ldots (1).$$

Also $HM=y$, $=h\cot\theta+h\cot[\theta+\alpha]$ $\ldots\ldots\ldots\ldots (2)$.

Let $z=$semi-minor axis, $=$ordinate in circular section through S, middle point of HK. Let $r=$radius of section.

Then $z=\sqrt{r^2-\overline{TS}^2}$, $=\sqrt{r^2-[(x/z)-y]^2\sin^2\theta}$ $\ldots\ldots\ldots\ldots (3)$,

since $\angle TMS=\angle ONR$, $=\theta$. Also,

$$r=NT\tan\alpha, \ =\left[\frac{h}{\sin\theta}+\left(\frac{x}{2}-y\right)\cos\theta\right]\tan\alpha \quad \ldots\ldots\ldots (4).$$

Volume $NHK=[\frac{1}{3}\pi]h[x/2]$, $=C$ $\ldots\ldots\ldots\ldots (5)$.

Let $\cot\theta=\beta$, $\cot\alpha=k$. (1) becomes, $x=\dfrac{2hk[\beta^2+1]}{k^2-\beta^2}$ $\ldots\ldots\ldots\ldots (6)$;

(2) becomes, $y=\dfrac{h[\beta^2+1]}{k+\beta}$ $\ldots\ldots\ldots (7)$, and $\dfrac{x}{2}-y=\dfrac{h\beta\,[\beta^2+1]}{k^2-\beta^2}$ $\ldots\ldots\ldots (8)$.

From (4) by (8), $r=\dfrac{hk\sqrt{\beta^2+1}}{k^2-\beta^2}$ $\ldots\ldots\ldots\ldots\ldots\ldots (9)$.

Substituting in (3),

$$z=\sqrt{\frac{h^2k^2[\beta^2+1]}{[k^2-\beta^2]^2}-\frac{h^2\beta^2[\beta^2+1]}{[k^2-\beta^2]^2}}, \ =h\sqrt{\frac{\beta^2+1}{k^2-\beta^2}} \quad \ldots\ldots\ldots (10).$$

(5) becomes, $[\frac{1}{3}\pi]h\times\dfrac{hk[\beta^2+1]}{k^2-\beta^2}\times h\sqrt{\dfrac{\beta^2+1}{k^2-\beta^2}}=[\frac{1}{3}\pi]h^3k\left(\dfrac{\beta^2+1}{k^2-\beta^2}\right)^{\frac{3}{2}}=C$;

whence $h=\sqrt[3]{\dfrac{3C}{\pi k}}\sqrt{\dfrac{k^2-\beta^2}{\beta^2+1}}$ $\ldots\ldots\ldots\ldots\ldots (11)$.

Now $AB = h - l\sin\theta + \frac{1}{2}a\cos\theta$, $= \sqrt[3]{\frac{3C}{\pi k}}\sqrt{\frac{k^2-\beta^2}{\beta^2+1}} - \frac{l}{\sqrt{\beta^2+1}} + \frac{a\beta}{2\sqrt{\beta^2+1}}$,

$$= \frac{\sqrt[3]{\frac{3C}{\pi k}}\sqrt{k^2-\beta^2} - l + \frac{a\beta}{2}}{\sqrt{\beta^2+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (12),$$

in which β is the only variable. $dAB/d\beta =$

$$\frac{\sqrt{\beta^2+1}\left(-\sqrt[3]{\frac{3C}{\pi k}}\frac{\beta}{\sqrt{k^2-\beta^2}} + \frac{1}{2}a\right) - \left(\sqrt[3]{\frac{3C}{\pi k}}\sqrt{k^2-\beta^2} - l + \frac{a\beta}{2}\right)\frac{\beta}{\sqrt{\beta^2+1}}}{\beta^2+1} = 0$$

Equating to zero, and clearing of fractions,

$$(\beta^2+1)\left(-\sqrt[3]{\frac{3C}{\pi k}}\frac{\beta}{\sqrt{k^2-\beta^2}} + \frac{1}{2}a\right) - \left(\sqrt[3]{\frac{3C}{\pi k}}\sqrt{k^2-\beta^2} - l + \frac{a\beta}{2}\right)\beta = 0,$$

$$\text{or } \sqrt[3]{\frac{3C}{\pi k}}\left(\frac{\beta[\beta^2+1]}{\sqrt{k^2-\beta^2}} + \beta\sqrt{k^2-\beta^2}\right) = \frac{1}{2}a + l\beta.$$

Squaring and clearing, $[3C/\pi k]^{\frac{2}{3}}\beta^2[1+k^2]^2 = [\frac{1}{2}a + l\beta]^2[k^2-\beta^2]$. Whence,

$$l^2\beta^4 + al\beta^3 + \{[3C/\pi k]^{\frac{2}{3}}[1+k^2]^2 + \tfrac{1}{4}a^2 - l^2k^2\}\beta^2 - alk^2\beta - \tfrac{1}{4}a^2k^2 = 0 \ldots\ldots \ldots (13).$$

Now $l = \frac{1}{2}a\cot\alpha$, $= \frac{1}{2}ak$; also, $k = \{c/\frac{1}{2}[b-a]\}$, $= \{2c/[b-a]\}$ . . . . . (14).

$$\therefore l = \frac{ac}{b-a}. \quad \text{Also, } C = \pi\left(\frac{2a+b}{6}\right)^2[l + \tfrac{1}{3}c], = \pi\frac{[2a+b]^3c}{108[b-a]}.$$

$$\therefore 3C/\pi k = \{[2a+b]^3/216\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (16).$$

Substituting (14), (15), (16) in (13),

$$\frac{a^2c^2}{[b-a]}\beta^4 + \frac{a^2c}{b-a}\beta^3 + \left(\frac{[2a+b]^2\{[b-a]^2+4c^2\}}{36[b-a]^2} + \tfrac{1}{4}a^2 - \frac{4a^2c^4}{[b-a]^4}\right)\beta^2 - \frac{4a^2c^3}{[b-a]^3}\beta - \frac{a^2c^2}{[b-a]^2} = 0 \quad \ldots\ldots (17).$$

By solving this for β we get the maximum values of AB, provided (Fig. 1) H does not pass E. In Fig. 2, representing this condition $\angle EAB = \theta$,

$\therefore AB = a\cos\theta$.

Accordingly (17) will produce critical values of θ, provided $\cos\theta$ is not $> AB/a$, AB to be determined from (12).

It is evident that for any position of HK which cuts EA, the value of AB will be greater than that determined by the supposition made above. We must seek for maxima in this case by a different method.

Fig. 2.

DKA (Fig. 3) represents section of volume of water.

$$\text{Volume}=[\pi c/9]\{[\tfrac{1}{4}a^2+[\tfrac{1}{3}a+\tfrac{1}{6}b]^2+\tfrac{1}{2}a[\tfrac{1}{3}a+\tfrac{1}{6}b]\},$$

$$=[\pi c/324][19a^2+7ab+b^2] \quad \ldots\ldots\ldots\ldots(18).$$

Equation (1)—(4) and (6)—(10) apply here as in Fig. 1.

Now volume ADK=cone NDK—cone NDA,$=\frac{1}{3}h\times$[area elliptical segment DK]$-\frac{1}{3}l\times$[area circular segment AD] $\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(19)$.

Let $AB=s$, $\angle DAC=\theta$, $\angle BKA=\theta-\alpha$.

$$DK=DB+BK,=s\tan\theta+s\cot[\theta-\alpha],=s\left(\frac{1}{\beta}+\frac{k\beta+1}{k-\beta}\right),\ =\left(\frac{sk[1+\beta^2]}{\beta[k-\beta]}\right) \quad \ldots.(20).$$

$$\text{Area segment } DK=\frac{\pi zx}{2}-\frac{zx}{2}\left(\cos^{-1}\frac{[DK-\frac{1}{2}x]}{\frac{1}{2}x}-\frac{DK-\frac{1}{2}x}{\frac{1}{2}x}\sqrt{1-\left(\frac{DK-\frac{1}{2}x}{\frac{1}{2}x}\right)^2}\right.$$

Substitute from (6), (10), and (20):

$$\text{Area } DK=\frac{h^2k[\beta^2+1]^{\frac{3}{2}}}{[k^2-\beta^2]^{\frac{3}{2}}}\left[\pi-\cos^{-1}\left(\frac{s[k+\beta]}{h\beta}-1\right)\right.$$

$$\left.+\left(\frac{s[k+\beta]}{h\beta}-1\right)\sqrt{\frac{s[k+\beta]}{h\beta}\left(2-\frac{s[k+\beta]}{h\beta}\right)}\right] \quad \ldots\ldots(21).$$

Fig. 3.

Now $s=h-l\sin\theta+\frac{1}{2}a\cos\theta$.

$$\therefore\ h=s+l\sin\theta-\tfrac{1}{2}a\cos\theta,\ =s+\frac{2l-a\beta}{2\sqrt{\beta^2+1}}\ldots\ldots(22).$$

$$AD=s\sec\theta,=\frac{s\sqrt{\beta^2+1}}{\beta}.$$

$$\text{Area segment } AD=\tfrac{1}{4}a^2\left[\pi-\cos^{-1}\left(\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right)\right.$$

$$\left.+2\left(\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right)\sqrt{\frac{s\sqrt{\beta^2+1}}{a\beta}\left(1+\frac{s\sqrt{\beta^2+1}}{a\beta}\right)}\right]\ldots\ldots(23).$$

Substitute from (18), (21), (22), and (23) in (19),

$$\frac{k[s\sqrt{\beta^2+1}+l-\frac{1}{2}a\beta]^3}{3[k^2-\beta^2]^{\frac{3}{2}}}\left\{\pi-\cos^{-1}\left[\frac{s[k+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{1+\beta^2})]\}}-1\right]\right.$$

$$+\left[\frac{s[+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{\beta^2+1})]\}}-1\right]\times$$

$$\left.\sqrt{\frac{s[k+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{\beta^2+1})]\}}\left[2-\frac{s[k+\beta]}{\beta\{s+[(2l-a\beta)/(2\sqrt{\beta^2+1})]\}}\right]}\right\}$$

$$-\frac{a^3k}{24}\left\{\pi-\cos^{-1}\left[\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right]\right.$$

$$\left.+2\left[\frac{2s\sqrt{\beta^2+1}}{a\beta}-1\right]\sqrt{\frac{s\sqrt{\beta^2+1}}{a\beta}\left[1-\frac{s\sqrt{\beta^2+1}}{a\beta}\right]}\right]=\tfrac{3}{3}\tfrac{1}{2}\tfrac{1}{4}\pi c[19a^2+7ab+b^2],$$

which equation contains only s, β, and constants. However, the chance of solving it after differentiation seems extremely slight.

PROBLEMS FOR SOLUTION.

MISCELLANEOUS.

56. Proposed by S. HART WRIGHT, A. M., M. D., Ph. D., Penn Yan, N. Y.

In latitude 40° N.$=\lambda$, when the moon's declination is 5° 23′ N.$=\delta$, and the sun's declination 9° 52′ S.$=-\delta'$, how long after sunset will the cusps of the moon's crescent set synchronously, the moon having recently passed its conjunction with the sun ?

57. Proposed by GEORGE LILLEY, Ph. D., LL. D., Professor of Mathematics in the Oregon State University, Eugene, Oregon.

A particle is placed very near the center of a circle, round the circumference of which n equal repulsive forces are symmetrically arranged ; each force varies inversely as the mth power of its distance from the particle. Show that the resultant force is approximately $\frac{m_1 n(m-1)}{2r^{m+1}}\times CP$, and tends to the center of the circle, where m_1 is the mass of the particle, CP its distance from the center of the circle, and r the radius of the circle.

EDITORIALS.

The credit of preparing the index for this volume is due Editor Colow.

Dr. Artemas Martin, of the U. S. Coast and Geodetic Survey, has been

promoted to Chief of the Library and Archives Division, at a salary of $1800 per annum, the promotion taking effect July 1, 1897. Dr. Martin has just been elected a member of the "Circolo Matematico di Palermo," Italy.

We regret to announce the death of Prof. B. F. Burleson, which occurred at his home, in Oneida Castle, New York, on December 2. Mr. Burleson was born in Stockbridge, July 7, 1835, but had resided in Oneida Castle for many years, where he was highly esteemed. For a number of years he occupied the position of Principal of the Union School, and in this position he proved a most successful and acceptable teacher. He was extremely fond of mathematics and was very expert in solving difficult problems. For many years he was a frequent contributor to most of the mathematical journals published in this country, and enjoyed a wide acquaintance with well-known mathematical teachers in various parts of the country. Five years ago he was stricken with paralysis and had since suffered from several strokes, which was the final cause of his death. Mr. Burleson was one of those promising but unfortunate men who possessed only the advantages of a common school education. His knowledge of mathematics was obtained by self application and in this way he became a very able analyzer of difficult mathematical problems as his solutions of many difficult problems will show. Had he possessed the advantages of a mathematical course in one of our leading universities, his influence would undoubtedly have been felt in a larger way. All honor is due him for what he made of the opportunities he possessed and the advantages afforded him. There survive him, his widow, a daughter, and one son, George Burleson, of Buffalo, and a sister residing at Oneida Castle.

BOOKS AND PERIODICALS.

Elements of the Differential and Integral Calculus. By William S. Hall, E. M., C. E., M. S., Professor of Technical Mathematics in Lafayette College. 250 pages. Price, $2.25. (1897). New York: D. Van Nostrand.

Great activity has been displayed within the last year in the production of texts on the subject of the Calculus. Among the more recent books on this subject, Professor Hall's treatise is entitled to very favorable consideration. The two branches of the Calculus are treated together to great advantage. The formulas for differentiation are established by the method of limits, but the method of infinitesimals is also explained, and the differential notation used when there is advantage gained by it. The numerical problems illustrating the text and showing applications in engineering practice is an excellent feature of the book. The table of integrals for convenience of reference is more extended than is usual in books of the same scope. Throughout the work there is a great compactness both in the methods and form of treatment, and we find more subjects presented than in most of the elementary texts. The chapter on Differential Equations is one of the best features of the book.

The Calculus for Engineers, with Applications to Technical Problems. By Professor Robert H. Smith. Pages 176. Price, $3.00. 1896. London: Charles Griffin and Company. Philadelphia: J. B. Lippincott Company.

The aim of this treatise is to introduce the student at once to the more important uses of the Integral Calculus, and incidentally to those of the Differential Calculus. The development of the *rattionale* of the subject is based on essentially *concrete* conceptions. Considerable use is made of the graphic method where admissable. The effort has been made to make the treatment less formal than usual, and the meaning and use of results is illustrated by many applications to mechanics, thermodynamics, electrodynamics, problems in engineering design, etc. One of the most distinctive and important features of the book is the very complete and extended Classified Reference Tables of Integrals and Methods of Integration, which occupy 42 pages. The chapter on the integration of Differential Equations will prove an important aid in pointing out methods of dealing with various classes of problems. The book has some practical features that will especially recommend it to engineers and physicists. J. M. C.

The Tutorial Trigonometry. By William Briggs, M. A., F. R. A. S., and G. H. Bryan, Sc. D., F. R. S. London: W. B. Clive. New York: Hinds & Noble. Pages 326. Price, $1.00. 1897.

This latest issue in the series of Tutorial texts is a very satisfactory book. The definitions of the trigonometric functions is wisely introduced early. Most of the articles are written with commendable clearness, and it is only in minor points that we have noticed any defects or inaccuracies in the book. The chapter on the ambiguous case in the solution of triangles is especially clearly and concisely stated. The large number of well-chosen examples attached to each chapter add much to the completeness of the book for class use. The relative importance of subject-matter is indicated by the use of different type, which somewhat mars the appearance of the printed page, but this is a slight objection as compared with the advantage gained in clearness and in effective presentation of the subject to students. While very much after the order of the long list of trigononometries now in use, this book seems to cover about the right ground and bears the marks of a well-constructed text-book. J. M. C.

Regular Points of Linear Differential Equations of the Second Order. By Maxime Bôcher, Ph. D., Assistant Professor of Mathematics in Harvard University. Pages 23. 1896. Cambridge: Harvard University Press.

This excellent little treatise is intended quite as much for students of mathematical physics who may not be able to carry the subject further than is here done as for those intending to make a more extended study of the modern theory of linear differential quotations. J. M. C.

Past and Present Tendencies in Engineering Education. By Mansfield Merriman, Professor of Civil Engineering, Lehigh University, South Bethlehem, Pennsylvania.

This pamphlet of 17 pages, reprinted from Volume IV of the Proceedings of the Society for Promotion of Engineering Education, contains the instructive presidential address of Professor Merriman before that society, at its meeting on August 20, last.

Macfarlane on Discharge of Condenser. This pamphlet contains the interesting discussion of Dr. Macfarlane's paper, which was presented at the meeting of the American Institute of Electrical Engineers in May last, in which Mr. Steinmetz, Dr. Kennelly, and Dr. Perrine took part, and also the communicated reply of Dr. Macfarlane.

Numerical Problems in Plane Geometry. By J. G. Estill, of the Hotchkiss School, Lakeville, Conn. 144 pages. 1897. New York: Longmans, Green & Co.

These problems are meant to be used with other geometries. The book contains a graded set of problems on the five books of geometry, as the division into Books is generally made. The use of the metric system is begun at the very first. The problems, and the entrance papers in the latter part of the book, seem to have been selected with great care and excellent judgment. The discussion of logarithms, and the explanation of their use, and the use of the table, have been clearly made. In as much as some knowledge of the metric system and the ability to solve numerical problems in plane geometry is now required for admission to most colleges, this little treatise should be especially acceptable to preparatory schools. J. M. C.

Euclid: Books I.—IV. By Rupert Deakin, M. A., Headmaster of King Edward's Grammar School, Stourbridge. Price, 70 cents. 1897. London: W. B. Clive. New York: Hinds & Noble.

This edition of Euclid was prepared for the well-known "Tutorial Series." The notes at the end of each book supply excellent comments upon and analysis of the propositions, especially aiding the student to group together propositions in which similar methods of proof are used. Special care has been taken to encourage the working of "riders," and a section is given in which methods of attack are suggested, while exercises on the various methods have been interspersed throughout the text. The book is attractively printed, and should furnish an important aid in teaching elementary Euclid. J. M. C.

School Geometry. By J. Fred Smith, A. M., Principal of Iowa College Academy. 320 pages. 1897. Chicago: Scott, Foresman & Co.

While there is no special novelty or marked improvement in this on other text-books of like purpose and scope, yet it is well written and has several good features. The subject is approached gradually, and as far as may be the abstract through the concrete; it is more elementary than many of the books in common use, and in the earlier part separate figures indicate the successive steps of a construction instead of one figure for all the steps combined. The equation is early introduced and frequently used. Emphasis is placed upon the importance of original work, and a large number of theorems and problems are given as additional exercises. The side references, usual in other books, showing the authority for each step in a demonstration are omitted, but we doubt if this feature will be much in its favor with most teachers. The book is well printed, but not very neatly nor substantially bound. J. M. C.

Infallible Logic. A Visible and Automatic System of Reasoning. By Thomas D. Hawley, of the Chicago Bar. 8vo. 660 pages. Full Leather Binding. Price, $5.00. Chicago: The Dominion Publishing Co.

Standard books are ever welcome when they come to us in forms and bindings representing all the embellishments of the art of bookmaking. Such a book is *Infallible Logic* published by The Dominion Company, Chicago, a copy of which has just come to our desk. The contents are well arranged, the illustrations are fine, the print is clear and neat, and the binding is superb. The Dominion Company is forging ahead as the leading western publishing house making a specialty of fine subscription books. Having salespeople in nearly every nook of the country, the company enjoys a large and growing trade. As this company has a known reputation for liberality towards its agents and fair treatment of them, an agency in this community for the above book, or some other published by this company, would be a source of considerable profit to the one fortunate enough to secure it. Interested readers should write the company for full particulars.

Elementary Arithmetic. By William W. Speer, Assistant Superintendent of Schools, Chicago. 314 pages. 1897. Boston : Ginn & Company.

To the first book of this series we have previously directed special attention. The author emphasizes the importance of early bringing into view the definite relations of quantity. The idea of relative magnitude is made the basis of treatment in this new series of books. Hence simple ratios are made the key to the solution of all problems. The treatise is sufficiently different from others of a similar purpose to give it field for trial, and in the hands of *competent* teachers we predict it will give profitable results.

J. M. C.

The American Monthly Review of Reviews. An International Illustrated Monthly Magazine. Edited by Dr. Albert Shaw. Price, $2.00 per year in advance. Single numbers, 25 cents. The American Monthly Review of Reviews Co., 13 Astor Place, New York.

The January number of the *American Monthly Review of Reviews* is one of the best issues in the history of that magazine. From cover to cover it is thoroughly "live," alert, and forceful. The opening editorial department of "The Progress of the World" gives a clear and exhaustive New Year's summary of political conditions in both hemispheres at the threshold of 1898. The elaborate article on "The Future of Austria-Hungary," by an Austrian, is by all odds the best account yet given in the English language of the warring forces which threaten to undermine the dual monarchy of central Europe; Mr. Charles A. Conant's clean-cut analysis of the present demands for currency reform in the United States is something that no practical man of affairs should fail to read; Dr. W. H. Tolman's summing up of the municipal progress of New York City under Mayor Strong is just what is needed at this time as an encouragement of efforts for civic betterment everywhere; Lord Brassey's remarkable paper on "The Position of the British Navy," with Assistant Secretary Roosevelt's comments, is full of food for thought when read in connection with the compact digest of the United States annual naval report, which follows, and the review of Captain Mahan's new book; two noteworthy letters of Count Tolstoi on the doctrines of Henry George, one addressed to a German disciple of George and the other to a Siberian peasant, are also published in this number. Besides these important and spirited special features, the magazine's regular departments of "Current History in Caricature," "Leading Articles of the Month," "Periodicals Reviewed," and "New Books" cover such timely topics as Hawaiian annexation and the great strike in England.

The Cosmopolitan. An International Illustrated Monthly Magazine. Edited by John Brisben Walker. Price, $1.00 per year in advance. Single number, 10 cents. Irvington-on-the-Hudson.

Among the leading articles in the January number are the following: Stephen Girard and His College; The Real Klondike; Harold Frederick's "Gloria Mandi"; and A Brief History of our late War with Spain.

The Arena. An Illustrated Monthly Magazine. Edited by John Clark Ridpath, LL. D. Price, $2.50 per year in advance. Single number, 25 cents. Boston : The Arena Co.

The Open Court. A Monthly Magazine devoted to the Science of Religion, the Religion of Science, and the Extension of the Religious Parliament Idea. Edited by Dr. Paul Carus ; T. J. McCormack, Assistant Editor ; E. C. Hegeler, and Mary Carus, Associate Editors. Price, $1.00 per year in advance. The Open Court Publishing Co., Chicago, Ill.

The following periodicals have been received : *Journal de Mathématiques Élémentaires*, (1er Decembre 1897) ; *American Journal of Mathematics*, (October, 1897) ; *The Mathematical Gazette*, (October, 1897) ; *L'Intermédiare des Mathématiciens*, (Novembre 1897) ; *Miscellaneous Notes and Queries*, (October, 1897) ; *The Kansas University Quarterly*, (October, 1897) ; *The Monist*, (October, 1897) ; *The Educational Times*, (December, 1897) ; *Science*, (Nos. for the year to September 24, 1897) ; *Bulletin of the American Mathematical Society*, (November, 1897) ; *The Ohio Teacher*, (November, 1897).

THE AMERICAN MATHEMATICAL MONTHLY'S Clubbing List :

	REGULAR PRICE.	WITH MONTHLY.
The American Monthly Review of Reviews	$2 50	$4 00
The Forum	3 00	4 50
The Cosmopolitan	1 00	2 85
The Arena	3 00	4 50
The Century	4 00	5 50
St. Nicholas	3 00	4 50
Popular Astronomy	2 50	4 25
Atlantic Monthly	4 00	5 00
The Critic (weekly)	3 00	5 00
The Outlook	3 00	4 75
The Ohio Educational Monthly	1 50	3 50
American Journal of Education	1 00	2 50
McClure's	1 00	2 80
Mathematical Magazine (quarterly)	1 00	3 00
Annals of Mathematics (bi-monthly)	2 00	4 00

Other magazines not mentioned above may be obtained from us at reduced rates.

B. F. FINKEL, J. M. COLAW, Editors.

SOME ERRATA IN No. 11.

Page 282, line 2, for "Ay" read Ay^2.

Page 282, line 17, for "$(2x-m^2$" read $(2x-m)^2$.

Page 283, line 5, for "$+b$" read $+b^2$.

Page 286, supply letter D in the figure.

Page 286, line 14, insert *will* before *OED*.

Page 286, line 5 from bottom, for "$\frac{1}{2}h$" read $\frac{1}{4}h$.

Page 287, line 17, for "27" read 27/2.

Page 288, line 2, for "R^3" read R^2.

Page 288, line 11, for "$\int_{-\sqrt{r^2-x^2}}^{-\sqrt{r^2+x^2}} zdy$" read $\int_o^r dx \int_{-\sqrt{r^2-x^2}}^{+\sqrt{r^2-x^2}} zdy$.

Page 288, line 12, for "$\int_o^s$" read $\int_o^r$.

Page 288, line 18, for "OF" read OB.

Page 288, line 19, for "$(2br/x)$" read $(2br/h)$.

Page 288, line 20, for "O" read 0.

Page 290, last line, for "sume" read sums.

INDEX.

ALGEBRA (See Solutions and Problems).
ARITHMETIC (See Solutions and Problems).
BIOGRAPHIES (With Portraits)—
DeMorgan, Augustus, by Dr. George Bruce Halsted 1–5
Hoüel, Guillaume Jules, by Dr. George Bruce Halsted..... 99–101
Newton, Hubert Anson, by Prof. Andrew W. Phillips..... 67–71
Sylvester, James Joseph, by Dr. George Bruce Halsted..... 159–168
Vasiliev, Alexander Vasilievitch, by Dr. George Bruce Halsted..... 265–267
Wood, DeVolson..... 197–199
Euler, Leonhard, by B. F. Finkel..... 297–302
BOOKS, Reviewed, 33-34, 64-65, 98, 124-127, 158, 195-196, 231-234, 259-263, 294-296, 324-328.
Elements of Analytical Geometry of Two Dimensions, by Briot and Bouquet,—translated and edited by James Harrington Boyd, 33; *The Outlines of Quaternions*, by Lieutenant-Colonel H. W. L. Hime, 34; *Plane Surveying*, by William G. Raymond, 34; *Determinants*, by J. M. Taylor, 64; *Elements of Theoretical Physics*, by Dr. C. Christiansen,—translated by W. F. Magie, 65; *Principles of Mechanism*, by Stillman W. Robinson, 65; (1) *Macauley's Essay on Milton*, (2) *Shakespeare's Midsummer Night's Dream*, (3) *Scott's Woodstock*, (4) *Milton L'Allegro, Il Penseroso, Comus, Lycidas*, (5) *"George Elliot's" Silas Marner*, issued by American Book Company, 65; *An Elementary Treatise on Plane Trigonametry*, by E. W. Hobson and C. M. Jessop, 65; *Composite Geometrical Figures*, by George A. Andrews, 98; *National Geographic Monographs*: (1) *Physiographic Processes*, by J. W. Powell, (2) *Physiographic Regions of the U. S.*, by J. W. Powell, (3) *Lakes and Sinks of Nevada*, by I. C. Russell, (4) *Mt. Shasta*, by J. S. Diller, 98; *Algebra Reviews*, by Edward Rutledge Robbins, 124; *Thoughts on Religion*, by the late George John Romanes,—edited by Charles Gore, 124; *Darwin, and After Darwin*, by George John Romanes, 124; *University Algebra*, by C. A. Van Velzer and Chas. S. Slichter, 125; *Text-Book on Dynamics*, by William Briggs and G. H. Bryan, 125; *Theoretical Mechanics: Fluids*, by J. Edward Taylor, 125; *A Treatise on Elementary Hydrostatics*, by John Greaves, 125; *Geometry of the Similar Figures and the Plane*, by C. W. C. Barlow and G. H. Bryan, 126; *Modern Plane Geometry*, by G. Richardson and A. S. Ramsey, 126; *Our Notions of Number and Space*, by Herbert Nichols, assisted by W. E. Parsons, 126; *Business: Forms, Customs, and Accounts*, by Seymour Eaton, 126; *Spencerian System of Penmanship: Common School Course:* No. 10,—*Connected Business Forms*, No. 11,—*Double Entry Bookkeeping*, 126; *Patriotic Citizenship*, by Thomas J. Morgan, 127; *Elementary Lessons in Algebra*, by Stewart B. Sabin and Charles D. Lowry, 127; *Elements of Plane Geometry*, by John Macnie,—edited by Emerson E. White, 127; *Inductive Studies in English Grammar*, by William R. Harper and Isaac M. Burgess, 127, *Theory of Discrete Manifoldness*, by F. W. Franklin, 158; *Recent Books on Quaternions*, by Dr. Alexander Macfarlane, 158; *Application of Hyperbolic Analysis to the Discharge of a Condenser*, by Dr. Alexander Macfarlane, 158; *Introduction to American Literature*, by F. V. N. Painter, 158, *Differential Equations*, by D. A. Murray, 195; *Analytical Geometry*, by F. R. Bailey and F. S. Woods, 196; *Higher Algebra*, by George Lilley, 196; *The Non-Regular Transitive Substitution Groups whose Order is the Product of Three Unequal Prime Numbers*, by Dr. G. A. Miller, 231; *A History of the United States*,

by Allen C. Thomas, 231; *The Tutorial Statics*, by William Briggs and G. H. Bryan, 231; *Grammar School Arithmetic by Grades*, by Eliakim Hastings Moore, 232; *Elementary Text-Book of Physics*, by Prof. Wm. A. Anthony and Cyrus F. Brackett,—revised by Prof. William Francis Magie, 232; *Theory of Physics*, by Joseph S. Ames, 232; *The New Arithmetic:* Part One for Teachers, by William W. Speer, 232; *Mathematical Questions and Solutions*, from the "Educational Times," 233; *Descriptive Geometry: Straight Line and Curves*, by William J. Meyers, 233; *Introduction to Infinite Series*, by William F. Osgood, 233; *Intermediate Algebra*, by William Briggs and G. H. Bryan, 233; *Elementary and Constructional Geometry*, by Edgar H. Nichols, 233; *The Science of Mechanics*, by Dr. Ernst Mach,—translated by Thomas J. McCormack, 234; *Elementary Mathematical Astronomy*. by C. W. C. Barlow and G. H. Bryan, 234; *A Text-Book of Light*, by R. Wallace Stewart, 259; *On the Transitive Substitution Groups that are Simply Isomorphic to the Symmetric or Alternating Group of Degree Six*, by Dr. G. A. Miller, 260; *New Principles of Geometry with Complete Theory of Parallels*, by Nicolái Ivánovish Lobachévski,—translated by Dr. George Bruce Halsted, 260; *A Text-Book of Physics*, by Edwin H. Hall and Joseph Y. Bergen, 260; *Ordinary Differential Equations*, by James Morris Page, 260; *On the Primitive Substitution Groups of Degree Fifteen*, by Dr. G. A. Miller, 261; *Higher Arithmetic*, by Wooster Woodruff Beman and David Eugene Smith, 261; *A Brief Introduction to the Infinitesimal Calculus*, by Irving Fisher, 261-263; *Plane and Solid Analytical Geometry*, by Frederick H. Bailey and Frederick S. Woods, 294; *Famous Problems of Elementary Geometry*, translated by Wooster Woodruff Beman and David Eugene Smith, 294; *Popular Scientific Lectures*. by Dr. Ernst Mach,—translated by Thomas J. McCormack, 294; *Field Manual for Railroad Engineers*, by J. C. Nagle, 294; *A Chapter in the History of Mathematics*, by W. W. Beman, 295; *Darwin and After Darwin: Part II., Post-Darwinian Questions*, by Dr. Romanes, 295; *The Probability of Hit when the Probable Error in Aim is Known, with a Comparison of the Probabilities of Hit by the Method of Independent and Parallel Fires from Mortar Batteries*, by Mansfield Merriman, 296; *Contributions to the Geometry of the Triangle*, by Robert J. Aley, 296; *Elements of the Differential and Integral Calculus*, by William S. Hall, 324; *The Calculus for Engineers, with Applications to Technical Problems*, by Prof. Robert H. Smith, 325; *The Tutorial Trigonometry*, by William Briggs and G. H. Bryan, 325; *Regular Pionts of Linear Differential Equations of the Second Order*, by Maxime Bocher, 325; *Past and Present Tendencies in Engineering Education*, by Mansfield Merriman, 325; *Macfarlane on Discharge of Condenser*, by Alexander Macfarlane, 325; *Numerical Problems in Plane Geometry*, by J. G. Estill, 326; *Euclid: Books I.—IV.*, by Rupert Deakin, 326; *School Geometry*, by J. Fred Smith, 326; *Infallible Logic*, by Thomas D. Hawley, 326; *Elementary Arithmetic*, by William W. Speer, 327.

DIOPHANTINE ANALYSIS (See Solutions and Problems).

EDITORIALS, 32-33, 63-64, 124, 194-195, 230-231, 259, 323-324.

ERRATA, 66, 236, 328.

GEOMETRY (See Solutions and Problems).

MATHEMATICAL PAPERS—

Aley, Robert J., A Device for Extracting the Square Root of Certain Surd Quantities 204-208

Burnham, A. C., On a Solution of the General Biquadratic Equation..... 243-245

On the Complex Roots of Numerical Equations of the Third and Fourth Degree 201-204

Calderhead, J. A., Equation of Payments 245–247
Calderhead, James A., and Benj. J. Yanney, New and Old Proofs of the Pythagorean Theorem 11–12, 79–81, 168–170, 250–251, 267–269
Gruber, M. A., Integral Sides of Right Triangles 106–108
Halsted, George Bruce, Non-Euclidean Geometry: Historical and Expository 10, 77–79, 101–102, 170–171, 200, 247–249, 269–270, 307–308
Heal, Wm. E., Some Divisibility Tests 171–172
Hopkins, G. I., Euclidean Geometry Without Disputed Axioms 251–255
Johnson, Edgar H., Concerning Conics through Four Points 104–105
Kellogg, Late Ausel N., Empirical Formulæ for Approximate Computation .. 35–49
Kummell, Charles H., Discussion of Merit Contests in College Examinations by the Method of Least Squares 129–132
Lilley, George, Zero, Infinitesimals, Infinity, and the Fundamental Symbol of Indetermination 255–259
Lovett, Edgar Odell, Sophus Lie's Transformation Groups .. 237–242, 270–275, 308–313
Lyle, John N., Multi-Directional Geometry 123
Macnie, John, Introduction to Differentiation 172–175
Miller, G. A., On the Solution of the Quadratic Equation 5–9, 71–77
Nicholson, J. W., A Simple Deduction of the Differential of $\log x$ 306
Philbrick, P. H., A Theorem on Prismoid 108
Roe, E. D., Jr., On the Circular Points at Infinity 132–145, 236
Two Developments 82–86
Stevens, M. C., A Method for Developing $\cos^n \theta$ and $\sin^n \theta$ 103–104
Yanney, Benj. F., and James A. Calderhead, New and Old Proofs of the Pythagorean Theorem 11–12, 79–81, 168–170, 250–251, 267–269
Zerr, G. B. M., Moments of Inertia 303–306

MECHANICS (See Solutions and Problems).

MISCELLANEOUS (See Solutions and Problems).

NOTES, 96-98, 122-123, 291-294.

International Congress of Mathematicians at Zurich in 1897, by Dr. George Bruce Halsted, 96–97; "The same old blunder," by Dr. Halsted, 97–98; Death of James Joseph Sylvester, 98; Notes on Problem 39, by Lewis Neikirk and J. B. Webb, 119–120; Note on Mr. Becher's Article in October (1896) number of MONTHLY, by J. R. Baldwin, 122; Note on Professor Lilley's Criticism (MONTHLY, Vol. III., No. 3) of the Solution IV. (Vol. II., page 190), by Wm. E. Heal, 122–123; Multi-Directional Geometry, by John N. Lyle, 123; Death of Prof. Ollis Howard Kendall, 124; Death of De Volson Wood, 231; Corrections and Revisions of the Article "On the Circular Points at Infinity," 236; The Irving Hopkins Fallacy, by George Bruce Halsted, 291; Note on Professor Hopkin's paper on "Euclidean Geometry without Disputed Axioms," by Benj. F. Yanney, 291; Note on Dr. Lilley's Article in the October number, by Prof. Milton L. Comstock, 292; Note on Dr. Lilley's Article, by Henry Heaton, 292; Note on the Same Article, by Editor Finkel, 293.

PORTRAITS, Facing pages 1, 67, 99, 159, 197, 237, 265.

DeMorgan, 1; H. A. Newton, 67; Hoüel, 99; Sylvester, 159; DeVolson Wood, 197; Zerr, Arnold, Heal, Drummond, Anthony, 237; Vasiliev, 265.

PERIODICALS, 34, 66, 127-128, 196, 234-236, 263-264, 296, 327-328.

SOLUTIONS AND PROBLEMS.

ARITHMETIC.

Account, A owes me \$100 due in 2 years, and I owe him \$200 due in 4 years, when can I pay him \$100 to settle equitably? No. 70 12–13

Account, if he owes me $200 and I owe him $100, when he can pay $100 to settle. No. 71 50
Account, same as No. 70, except that interest draws interest to time of settlement. No. 73 87
Bonds, how much more, quotatiotions given, was a $1000 U. S. bond worth in London than in Philadelphia. No. 74 ..... 108, 146
Cask, how much water in wine cask after drawing and supplying water, and then drawing mixture and adding wine. No. 80 176-177
Cattle, to find how many horses, cows and sheep a Texan owns. No. 78 .... 147, 210
Elephants, to divide an estate of 17, among three sons. No. 76 109
Field, length and quality given, to find worth. No. 72 50-51
Gold pieces, to find how many $20 pieces in a room. No. 79 175-176
Notes, what must be paid on certain date for three notes of same date, due at different times, to realize 8% compound interest on investment. No. 83 275-276
Sun, to find distance body will fall in first second, on the sun. No. 81 209
Train, time to count clicking of rails under, that number counted may equal speed of train in miles per hour. No. 77 146-147
Traveling, to find find distance traveled by each of two men. No. 82 209-210
Trench, men digging, for loaves, to find worth of barrel of flour, in problem in proportion. No. 75 109

ALGEBRA.

Algebraic equation, prove that $x^n \pm x^{n-1} + x^{n-2} \pm \ldots\ldots + (\pm 1)^{n-1}x + (\pm 1)^n = (x \mp 1)^n \pm A(x \mp 1)^{n-1} + B(x \mp 1)^{n-2} \pm \ldots\ldots + (\pm 1)^n x$. No. 69 53-54
Equation, in $\sqrt[3]{(a+x)} + \sqrt[3]{(a-x)} = \sqrt[3]{c}$, to find x. No. 70 147-149
Limit, to find limit of given expression, when $x=0$. No. 71 149-150
Prove that $[2\sqrt{2+\sqrt{3}}]/[4+\sqrt{6}-\sqrt{2}] = \sqrt{6}-\sqrt{2}+\sqrt{3}-2$. No. 72 210-211
Quadrilateral farm, inscriptible and circumscriptible, with area, etc., given, to find sides of farm. No. 75 315-318
Series, sum to n terms, $n\cos\theta + (n-1)\cos 2\theta + (n-2)\cos 3\theta$, etc. No. 68 51-53
Solve according to given conditions: $\sqrt{x+1} + \sqrt{x} = 3/\sqrt{1+x}$. No. 74 314-315
Worth, to find of each of 5 persons, under conditions given in the problem. No. 73 211-212

GEOMETRY.

Aley's problems involving proofs in the geometry of the triangle. Nos. 71, 73, and 74 ..... 177-178, 178-179
Card, to divide a square, into right-lined sections, that a given rectangle can be formed from them; likewise, a square from a rectangular card. No. 69 150-152
Carpet, to find length of piece that can be placed diagonally in room. No. 67 87-89
Circle, divided at $A, A_1, A_2, A_3 \ldots\ldots$ into n equal parts, has diameter through A and O joined with points of division, to prove that $OA_1 - OA_2 + OA_3 - OA_4 + \ldots\ldots \pm OA_{\frac{1}{2}(n=1)}$. No. 68 110-111
Curve, formula for area of given, to find (unsolved). No. 72 178
Ellipse, to find area of, which is isogonal conjugate to a given straight line with respect to given triangle. No. 77 214-215
Hyperbola, locus of points whose polars with respect to a given parabola touch circle of curvature at vertex is an equilateral. No. 66 15-16
Locus of ultimate intersections of plane, passing through $(0, 0, c)$ and touching circle $x^2 + y^2 = a^2$, $z=0$. No. 75 180-181
Normals to parabola, required number that can be drawn from any point. No. 78 276-278
Parabola, to prove that locus of center of circle which passes through vertex of a, and through its intersections with a normal chord is the parabola $2y^2 = ax - a^2$. No. 70 111-112

Quadrilateral, sides given, and diagonals equal, to find them, and also area. No. 43 56–56
Quadrilateral, to construct of given area, the diagonals given, cutting each other in given ratios, and at given angle. No. 79 278–279
Ratio of AO to OA', where O is center of inscribed circle, and AO produced meets circumcircle in A'. No. 74 179
Steiner's ellipse, axes of, are parallel to Simson's lines belonging to extremities of Brocard's Diameter. No. 65 13–15
Triangular piece of land, to find dimensions of, when lines from point within, to angles, etc., given. No. 76 212–213

CALCULUS.

Bucket, in form of frustum of cone. with diameters at base and top, and its perpendicular height, given, to find height to which water contained therein will rise when bucket is overturned. No. 62 319–323
Cardioid, to find length of closed curve of, its area, maximum longitudinal circumference of solid generated, if made to revolve about its axis, etc. No. 56 19–21
Curve, polar equation given, to find loops and area. No. 61 318–319
Differential equation solved. No. 59 216–217
Horse tethered by rope to post in circular fence around circular piece of ground, to find ground fed over. No. 55 16–19
Leaf of strophoid, in leaf of, find axis of inscribed leaf of lemniscate, and in leaf of lemniscate find axis of inscribed leaf of strophoid. No. 49 152–155
Method for finding locus of intersection of two lines, one of which passes through a fixed point and rotates uniformly about this point, and the other passes through a point which moves uniformly along the arc of a given circle, and rotates about this point. No. 58 ..... 57–59, 89–90
Solve the equation $(1+x^2)d^2y/dx^2-(2x)dy/dx+2y=0$. No. 57 56–57
Sphere—cone, to remove given part of sphere by conical hole, whose axis is that of sphere, and vertex at surface of same, and find height of cone and diameter of its base. No. 60 217–219

MECHANICS.

Ball, particle, motion of, at given latitude, if earth a perfect sphere with frictionless surface. No. 41 ..... 22, 59–60
Cannon-ball, projected upward with given velocity, to find height to which will rise and the time of flight. No. 45 113
Cassinian oval, to find law of the force that the orbit may be. No. 40 22
Chain, if a, of given length is hung over a smooth peg with one end higher than the other and let go, to find time chain will run off the pin. No. 52 279–281
Equilibrium, position of, of center of triangle, whose sides repulse a center of force with an intensity that varies inversely as the distance of center of force from each side. No. 44 ..... 112–113, 155–157
Hiawatha, to find how "swift of foot." No. 51 220–221
Lens, focus of convex surface of a plano-convex, which will converge parallel monochromatic rays to a given focus. No. 47 115
Locus of center of gravity, to find, of an arc of constant length for a parabola. No. 53 281
Quadrilateral, to find pressure of water upon in given problem, and the position of the center of mean pressure. No 50 219–220
Rings, two equal heavy, connected by string over a peg at focus of a conic, will be in equilibrium at all points on the curve. No. 48 116

Stick of timber, rectangular, of known dimensions, placed on platform, etc., when and in what manner will strike the ground. No. 49..... 181–183
Vibration of particle, to find time of, slightly displaced from center of a solid cylinder in direction of axis. No. 42..... 22–23
Weights, two rest on concave side of a parabola whose axis is horizontal, and are connected by a string over a peg at focus. No. 43..... 90–92
Woman—moon, how long for old lady to go up, "if tossed in a basket ninety times as high as the moon." No. 46..... 114–115

DIOPHANTINE ANALYSIS.

Field, to find sides and area of rectangular, enclosed with fence and subdivided by fences on both diagonals. No. 49..... 184
Least values, to find, of x and y in integers, in $x^2 - 114\frac{1}{4}y^2 = \mp 3$. No. 53.... 281–282
"Magic Square," prove that a, of 9 integral elements, whose rows, columns, and diagonals have a constant sum, is only possible when this sum is a multiple of three. No. 52..... 189
Number, if any positive integral, be divided by another, with remainder of one, then any positive integral power of first divided by second, will leave a remainder 1. No. 48..... 28–29
Parallelopiped, edges of a rectangular, within 1 of a given proportion, etc., to find integral values for x. No. 50..... 185–187
Series of values, find two, for x in integral terms of a and b, in the expression $2x^2 - 2ax + b^2$. No. 54..... 282–283
Sum of two consecutive integral squares equals a square, to find six sets of values. No. 47..... 24–28
Triangular square numbers, the difference between the roots of two successive triangular numbers, that are also square numbers, equals the sum of two successive integral numbers, the sum of whose squares will be a square number. No. 51..... 189–189

AVERAGE AND PROBABILITY.

Average length of chords drawn from every point of circumference in circle of given radius, through a point a given dist. from center, etc. No. 43 93, 117–118
Base ball, to find the probability of either team winning, when score at end of fifth inning is given. No. 45..... 118–119
Chance that sixteenth day will be cloudy, if clear for fifteen consecutive days. No. 55..... 284
Church, probability that A and B will be at, and C not, when previous attendance of each is given. No. 42..... 30
Chords, average length of all, that may be drawn from one extremity of the major axis of an ellipse, if drawn at equal angular intervals. No. 44..... 93–94
Circular field, to find chance that at least two of four men, starting from random points on circumference, at random courses, across it, will meet. No. 46..... 119
Desert, to find probability that a man at center of circular, will be off in a given time. No. 39..... 29, 60–61, 120
Dice, to find expectations of one of the players. No. 48..... 190
Ellipse, to find average length of the chords drawn by joining every point of, with every other point. No. 40..... 29–30
Ellipse, to find average length of the chords that may be drawn from one extremity of the major axis to every point of the curve. No. 47..... 119
Man at center of circle, when diameter equals three of his steps, to find the probability that he will step outside of circle at two, and also at three steps. No. 54..... 283–284

Mean area of divisions, when line drawn at random across the chord and given arc of circular segment. No. 41. 30
Notes on problem No. 39 119–120
Segment, find average volume of, cut off from sphere by plane. No. 57..... 192–193
Sentences, chances two pupils will translate words in, from Latin, in same order. No. 53 (unsolved) 63
Straight line, chance that no part greater than b, if one of given length is divided into three parts by two points taken at random. No. 52 221–222
Square and triangle, to find average arc common to both, if fastened together and thrown on floor at random. No. 49..... 191
Triangles—lines—chords, to find average area and length, respectively, in the given problem. No. 50 191–193

MISCELLANEOUS.

Centroid to give general proof that, determines that point from which sum of distance to all other points of a given area is the minimum. No. 49.... 285
Conical section, to find volume of a segment of a right cone, whose diameter and perpendicular is given, if section is parallel with perpendicular of cone and includes ¼ of its circumference at base. No. 52 287–289
Ecliptic, what points of, were rising, setting, and on the meridian, in given latitude and longitude, at given time. No. 47..... 226–227
Ellipse, to describe curve marked out by foci, if ellipse confined to one vertical plane is suspended from a fixed point in space, coincident with a movable point on its circumference. No. 41 32
Hanging bar, to describe and analyze curve traced on wall by two ends of, suspended at two ends by string over peg in perpendicular wall. No. 41.. 31–32
Locus of P, if base BC of triangle ABC is $2c$, sum of two sides $2a$, and BP always perpendicular to AB, and cuts AC in P. No. 46..... 225–226
Moon, in case of *mischance*, with what force cow jumping over, would have struck Her Lunar Majesty in the face. No. 48 227
Numbers, required several, each of which when divided by 10 leaves 9; by 9 leaves 8, etc. No. 45. 223–224
Path of moon, describe and compute actual of, in July and August, 1896. No. 50 (unsolved)..... 285
Path of squirrel, to find, when squirrel jumps from top of one tree to top of another of different height. No. 44..... 96
Shafts falling, to find time in which each of two cylindrical iron shafts, standing perpendicular at the sea level, fall in still air. No. 40 31
Stock dealer traveling and buying and selling stock, from data given to find how far he traveled from home the entire trip, cost of each head of stock and entire gain. No. 51..... 285–286
Triangle, to find, whose sides and median lines are commensurable. No. 42 94–95
Triangle, to find, if possible, a right angled, the bisectors of the acute angles of which can be expressed in integral whole numbers. No. 43 95
Volume of a segment of a right cone, to find when diameter and perpendicular of cone are given, if section parallel with perpendicular of cone and includes ¼ of its circumference at base. No. 52..... 287–289